NCTS Short Course

Riemann-Hilbert Method in Integrable Systems

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There will be 5 lectures. Each lecture takes about 2 hours, with outline as below.

Lecture I: Riemann-Hilbert Problems for Orthogonal Polynomials and Painlevé Equations

Abstract: This lecture will introduce the notion of Riemann-Hilbert problems in the theory of orthogonal polynomials and in the theory of Painlevé equations. We begin with orthogonal polynomials and develop the Fokas-Its'-Kitaev representation of them via a matrix Riemann-Hilbert problem. Then we discuss the representation of solutions of Painlevé equations via Riemann-Hilbert problems associated to the inverse monodromy problem for some linear differential equations. We develop isomonodromic Schlesinger transformations as a tool for generating families of solutions of Painlevé equations, focusing on the example of the rational solutions of Painlevé-II.

Lecture II: Basic Theory of Riemann-Hilbert Problems

Abstract: A Riemann-Hilbert problem is fundamentally a problem of complex analysis, a kind of boundary-value problem for the Cauchy-Riemann equations. However, as with many problems of elliptic partial differential equations, a Riemann-Hilbert problem can be recast as a singular integral equation. This lecture will highlight some of the key ideas of the connection between Riemann-Hilbert problems and integral equations, with emphasis on the small-norm setting and how to achieve it by deformation techniques.

Lecture III: More Theory of Riemann-Hilbert Problems

Abstract: (A continuation of Lecture II)

Lecture IV: Asymptotic Analysis of Orthogonal Polynomials via Steepest Descent

Abstract: We show how the Fokas-Its'-Kitaev Riemann-Hilbert problem for orthogonal polynomials with exponentially varying weights can be deformed by systematic steps reminiscent of the method of steepest descent for contour integrals. The deformed Riemann-Hilbert problem is simple enough that its structure suggests an approximate solution, known as a parametrix. The parametrix will be constructed explicitly with the help of elementary and special functions. Then by comparing the parametrix to the exact solution we will arrive at a Riemann-Hilbert problem of small-norm type (cf., Lectures II-III). Estimates on the solution of the latter problem yield explicit leading-order asymptotic formulae for the orthogonal polynomials and related quantities of interest in applications such as random matrix theory.

Lecture V: Rational Solutions of the Painlevé-II Equation via Steepest Descent

Abstract: We give a second application of the Deift-Zhou steepest descent method to accurately approximate the rational solutions of the second Painlevé equation in the limit of large degree.