Qualified Examination: Topology
February 2007

1. Let A be a subspace of complete metric space. Show that $\bar{A}$ (the closure of A) is compact if and only if A is totally bounded.

2. Show that a topological space is connected if and only if every non-empty proper subset has a non-empty boundary.

3. A continuous function r from a topological space X onto a subspace A of X is a *retraction* iff $r|A$ (the restriction of r on A) is the identity on A. The subspace A of X is then called a *retract* of X. Show that

   (i) A retract in a Hausdorff space is a closed set.

   (ii) A subset A of X is a retract of X if and only if every continuous function $f : A \to Z$ has an extension to a continuous function $F : X \to Z$.

4. A compact space X is *maximal compact* iff every strictly larger topology on X is non-compact. Show that a compact space is maximal compact if and only if every compact subset is closed.

5. Show that the continuous image of a first-countable space need not be first-countable; but the continuous open image of a first-countable space is first-countable; i.e., if $f : X \to Y$ is continuous and open and X is first-countable, then $f(X)$ is first-countable.