1. A chemical engineer is investigating the yield of a process. There are k process variables, $x_1, \ldots, x_k$, and each variable can be run at a low and a high level. Thus this engineer runs a 2^k full factorial design with n center points, and $x_i, i = 1, \ldots, k$ are defined on a coded scale from -1 to 1. To represent the results, this engineer decides to fit a main effects only model, say

$$y = \beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k + \varepsilon,$$

where ε is the random error. Find the relationship between the least-squares estimates of $\beta_i, i = 1, \ldots, k$ and main effect estimates. (10 points)

2. Find the least-squares estimator of β in the model $\mathbf{y} = \mathbf{X}\beta + \varepsilon$ subject to a set of equality constraints on β, say $T\beta = c$. (10 points)

3. Consider the single-factor fixed-effects analysis of variance with 3 treatments:

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}, i = 1, 2, 3, j = 1, \ldots, n,$$

where ε_{ij} is an normal error with zero mean and variance σ^2. This model can be expressed in terms of the regression model with two indicate variables.

(1) Find the least-squares estimates of these regression coefficients and show the relationship with estimates of μ and τ_i. (5 points)

(2) Show how this regression model could be used to test the hypothesis, $H_0 : \tau_1 = \tau_2 = \tau_3 = 0$, and write down the corresponding test statistic and its distribution under H_0. (10 points)

4. Consider a p-dimensional random vector, $\mathbf{U}$, with zero mean vector and covariance matrix, Σ. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$ be the eigenvalues of Σ, and $P_1, \ldots, P_p$ be the corresponding eigenvectors.

(1) Find a $p \times p$ matrix W such that the components of $\mathbf{Z} = W\mathbf{U}$ are uncorrelated and their variances are all equal to 1. (5 points)

(2) Let B be any $p \times 1$ vector such that $\|B\| = 1$. Show that

$$\max_{B \perp P_1, \ldots, P_{i-1}} Var(B'\mathbf{U}) = \lambda_i,$$

and "max" is attained when $B = P_i$. (5 points)

(3) Let E_p denote the p-dimensional Euclidean space. Show

$$\min_{S_{i-1}} \max_{\|B\|=1, B \perp S_{i-1}} Var(B'\mathbf{U}) = \lambda_i,$$

where S_{i-1} is a space of $(i-1)$ dimensions in E_p. (10 points)

5. Let $X = (X_1, \ldots, X_p)'$ come from a p-dimensional multivariate normal distribution with mean vector μ and covariance matrix Σ, and A be a $p \times p$ symmetric matrix.

   (1) Show that the covariance of X with $X'AX$ is

   $$Cov(X, X'AX) = 2\Sigma A\mu. \text{ (5 points)}$$

   (2) Prove that $X'AX$ and BX are distributed independently if and only if $B\Sigma A = \mathbf{0}$. (10 points)

6. Consider the simple regression model is

   $$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots \beta_k x_k + \varepsilon,$$

   where $E(\varepsilon) = 0$ and $Var(\varepsilon) = \sigma^2$. Suppose that $n > k$ uncorrelated observations are available, and let y_i denote the ith observed response and x_{ij} denote the ith level of regressor x_j. Here inverse matrix of $(\mathbf{X}_n'\mathbf{X}_n)$ is assumed to exist, where $\mathbf{X}_n$ is the model matrix of these n observations. Let $\beta = (\beta_0, \beta_1, \ldots \beta_k)'$ be the coefficient vector. Suppose the $(n+1)$th observation is obtained. Compute the covariance matrix of current least-squares estimator, $\hat{\beta}_{n+1}$, based on $(\mathbf{X}_n'\mathbf{X}_n)^{-1}$. (15 points)

7. Consider the 2_{IV}^{6-2} design with the generators $I = ABCE$ and $I = BCDF$.

   (1) Construct this design. (5 points)

   (2) Show the complete alias structure of this design. (10 points)