編號：____________________

問題	1: 10分	2: 10分	3: 10分	4: 10分	5: 10分	總分: 100分
得分						
問題	6: 10分	7: 10分	8: 10分	9: 10分	10: 10分	
得分						

請詳列計算和推導過程書寫於題目下方空白處，並將答案書寫於下方指定處。

1. A *parallel system* is one that functions as long as at least one component of it functions. A particular parallel system is composed of three independent components, each of which has a lifelength with an exponential(λ) distribution. The lifetime of the system is the maximum of the individual lifelengths. What is the distribution of the lifetime of the system?

2. (a) Prove that the χ^2 distribution is *stochastically increasing* in its degrees of freedom; that is, if $p > q$, then for any a, $P(\chi^2_p > a) \geq P(\chi^2_q > a)$, with strict inequality for some a.

   (b) Use the results of part (a) to prove that for any $\nu, kF_{k,\nu}$ is stochastically increasing in k.

   (c) Show that for any k, ν and α, $kF_{k,\nu} > (k-1)F_{\alpha,k-1,\nu}$. (The notation $F_{\alpha,k-1,\nu}$ denotes a level-α *cutoff point*

3. Let Y have the Cauchy distribution, $f_Y(y) = \frac{1}{\pi(1+y^2)}$, $-\infty < y < \infty$.

   (a) Show that $F_Y(y) = \frac{1}{2} + \tan^{-1}(y)/\pi$.

   (b) Show how to simulate a Cauchy(a, b) random variable starting from a uniform$(0, 1)$ random variable.

4. Let $X_1, \ldots, X_n$ be a random sample from *the inverse Gaussian distribution* with pdf

$$f(x|\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} e^{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}}, \quad 0 < x < \infty.$$

   (a) Show that the statistics

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad T = \frac{n}{\sum_{i=1}^{n}(\frac{1}{X_i} - \frac{1}{\overline{X}})}$$

   are sufficient and complete.

   (b) For $n = 2$, show that $\overline{X}$ has an inverse Gaussian distribution, $n\lambda/T$ has a χ^2_{n-1} distribution, and they are independent. (Schwarz and Samanta 1991 do the general case.)

The inverse Gaussian distribution has many applications, particularly in modeling of lifetimes.

5. Let $X_1, \ldots, X_n$ be iid observations from a location-scale family. Let $T_1(X_1, \ldots, X_n)$ and $T_2(X_1, \ldots, X_n)$ be two statistics that both satisfy

$$T_i(ax_1 + b, \ldots, ax_n + b) = aT_i(x_1, \ldots, x_n)$$

for all value of $x_1, \ldots, x_n$ and b and for any $a > 0$.

(a) Show that T_1/T_2 is an ancillary statistic.

(b) Let R be the sample range and S be the sample standard deviation. Verify that R and S satisfy the above condition so that R/S is an ancillary statistic.

6. Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$. Find the best unbiased estimator of σ^p, where p is a known positive constant, not necessarily an integer.

7. Let X be an observation from the pdf

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1; 0 \leq \theta \leq 1.$$

(a) Find the MLE of θ.

(b) Define the estimator $T(X)$ by

$$T(X) = \begin{cases} 2 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $T(X)$ is an unbiased estimator of θ.

8. The random variable X has pdf $f(x) = e^{-x},\ x > 0$. One observation is obtained on the random variable $Y = X^{\theta}$, and a test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ needs to be constructed. Find the UMP level $\alpha = .10$ test and compute the Type II Error probability.

9. Find a pivotal quantity based on a random sample of size n from a $N(\theta, \theta)$ population, where $\theta > 0$. Use the pivotal quantity to set up a $1 - \alpha$ confidence interval for θ.

10. Let X_1, ..., X_n be a random sample from a $\mathrm{n}(\mu, \sigma^2)$ population.

    (a) If μ is unknown and σ^2 is known, show that $Z = \sqrt{n}(\overline{X} - \mu_0)/\sigma$ is a Wald statistic for testing $H_0 : \mu = \mu_0$.

    (b) If σ^2 is unknown and μ is known, find a Wald statistic for testing $H_0 : \sigma = \sigma_0$.