一百零一學年度國立中山大學應用數學系博士班第二次資格考數理統計試題
注意事項：

1. 本試卷共六大題，配分都寫在各題目裡。

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1. Suppose that $X_1, \ldots, X_n$ is a random sample from the distribution on the positive numbers with density $\theta^2 x e^{-\theta x}$.
   (i) Calculate the expectation of $\frac{1}{X_1}$. (5%)
   (ii) Determine a minimum variance unbiased estimator of θ. (15%)
   (Hint: you can consider the estimator to be the form $\frac{a}{\bar{X}}$, where $\bar{X}$ is the sample mean of $X_1, \ldots, X_n$ and a is a constant. If you consider this form, you still need to prove it is the minimum variance unbiased estimator)

2. Let $\{Y_n : n > 1\}$, $\{Z_n : n > 1\}$, and $\{W_n : n > 1\}$ be independent sequences (i.e. the three sequences are independent of each other) of random variables where: $Y_1, Y_2, \ldots$ are independent and identically distributed with $E(Y_1) = 0$ and $Var(Y_1) = 1$;
   $Z_1, Z_2, \ldots$ are independent with $E|Z_n| = n^2 \log(n)$ for all $n \geq 1$; and
   $W_1, W_2, \ldots$ are independent Bernoulli random variables with $P(W_n = 1) = n^{-4} = 1 - P(W_n = 0)$ for all $n \geq 1$.
   Let $X_n = (1 - W_n)Y_n + W_n Z_n, n \geq 1$.
   Determine the limiting distribution of $n^{-1/2} \sum_{k=1}^{n} X_k$ when $n \to \infty$. (20%)

3. Consider the parametric model
$$X \sim N(\theta, \theta^2), \theta > 0.$$
   (i) Fine the most power α-level test of
$$H_0 : \theta = \theta_0 \qquad vesus \qquad H_a : \theta = \theta_1,$$
   where $\theta_1 > \theta_0$. (10%)
   (ii) Does a uniformly most powerful α-level test of
$$H_0 : \theta = \theta_0 \qquad vesus \qquad H_a : \theta > \theta_0,$$
   exist? If yes, give the most powerful test; if not, give the reason why there is no such test. (10%)

4. Let X_1 and X_2 be two (absolutely) continuous independent random variables with common density function f and cumulative distribution function F.
   Let $Y = |F(X_1) - F(X_2)|$ and $Z = min(F(X_1), F(X_2))$.
   Prove Y and Z have the same distribution. (10%)

5. (10%) Let $\{X_1, \ldots, X_n\}$ be a random sample from $N(\mu, \sigma^2)$. Let $\bar{X}$ and S^2 be the sample mean and sample variance of $X_1, \ldots, X_n$, respectively. Prove $\bar{X}$ and S^2 are independent. (Hint: You can consider using Basu's Theorem, but it is not necessary to use it). (10%)

6. (20%) Let $X_1, \ldots, X_n$ be a sequence of random variables satisfying
$$X_i = \theta X_{i-1} + \epsilon_i, i = 1, 2, \ldots, n,$$
   where $X_0 = 0$ and ϵ_i are i.i.d. $N(0, \sigma^2)$.
   Please derive the likelihood ratio test for
$$H_0 : \theta = 0 \qquad vesus \qquad H_a : \theta \neq 0$$
   You only need to derive the test statistic.