注意事項：

1. 本試卷共九大題，1-4題每10分，5-9題每12分。

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1. Suppose $X_1, \cdots, X_n$ are independent random variables, where for $i = 1, \cdots, n$, X_i has $\Gamma(\alpha_i, \beta)$ distribution, $\alpha_i, \beta > 0$. Let $T_i = \sum_{j=1}^{i} X_j, i = 1, \cdots, n$, and $Y_i = \frac{T_i}{T_{i+1}}, i = 1, \cdots, n-1$.
   (i) Show that Y_1, T_2 are independent, and find their distributions.
   (ii) Show that when $\alpha_i, i = 1, \cdots, n$ are known, T_n is a complete sufficient statistic for β.
   (iii) Show that $\mathbb{Y}_{n-1} = (Y_1, \cdots, Y_{n-1})$ is a ancillary statistic for β, and discuss whether $Y_1, \cdots, Y_{n-1}, T_n$ are independent, and find their joint distribution.

2. Let $X_n, n \geq 1$, have a negative binomial distribution $NB(n, \theta)$ starting from 0, where $0 < \theta < 1$.
   (i) Find a such that $X_n/n \xrightarrow[n \to \infty]{P} a$.
   (ii) Find the approximate value of $P(X_n \leq x), x \in R$, when n gets large.

3. In a chemical laboratory, the concentration levels of certain chemical of n kinds of liquid are measured. Each liquid is measured independently twice. Assume the accuracy of each measurement is the same. Then we have independent measurement samples $X_i, Y_i, i = 1, \ldots, n$, where X_i, Y_i have distribution $N(\mu_i, \sigma^2)$, with $\mu_i \in R$, and $\sigma^2 > 0$ being the unknown parameters.
   (i) Find the maximum likelihood estimates $\hat{\mu}_i, i = 1, \ldots, n$, and $\hat{\sigma}^2$, of $\mu_i, i = 1, \ldots, n$, and σ^2, respectively.
   (ii) Are $\hat{\mu}_i, i = 1, \ldots, n$, and $\hat{\sigma}^2$ consistent estimates? If they are, prove it; if they are not, explain the reason and find a consistent estimate of the corresponding parameter respectively.

4. Suppose that $(X_{i1}, X_{i2}), i = 1, \ldots, n$, are i.i.d. bivariate normal distribution with mean $\mu = (\mu_1, \mu_2)$, variances σ_1^2, σ_2^2, and correlation ρ. Let $\tau^2 = \text{Var}(X_{i2}|X_{i1})$.
   (i) Derive moment estimator (ME) of ρ and τ^2, respectively.
   (ii) Calculate the covariance matrix of $(X_{i1}^2, X_{i2}^2, X_{i1}X_{i2})$.
   (iii) Find the limiting distributions of the MEs in (i).

5. Let $X_1, \cdots, X_n$ be a random sample from either $Unif(\theta, \theta+1), \theta > 0$, distribution.
   (i) Find a sufficient statistic for parameter θ, and explain if it is a minimal sufficient statistic.
   (ii) Consider testing $H_0 : \theta = 0$. For each of the following alternatives, find the UMP(uniformly most powerful) or MP(most powerful) test of size α.
      (a) $H_1 : \theta \geq 1$.
      (b) $H_1 : \theta = \theta_1$, where θ_1 is a fixed alternative between 0 and 1, $0 < \theta < 1$.

6. Suppose $X_1, \ldots, X_n$ are independent $\mathcal{E}(\lambda)$, $\lambda > 0$, random variables. If we could not observe the values of $X_1, \ldots, X_n$, but the number N of $X_i, i = 1, \cdots, n$ less than or equal to a constant $M, M > 0$. i.e. $N = \sum_{i=1}^{n} I_{\{X_i \leq M\}}$. Let $p = P(X_1 \leq M) = 1 - e^{-\lambda M}$,
   (i) Find the maximum likelihood estimate (MLE) $\hat{p}$ of p, and its corresponding distribution function.
   (ii) Find the Cramér-Rao lower bound of p, and show that $\hat{p}$ is also a UMVUE of p.
   (iii) Find the MLE of λ.

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7. You are given a coin, which you are going to test for fairness. Let the probability of a head be p, and consider testing $H_0 : p = 1/2$ against $H_1 : p < 1/2$. You toss the coin until you observe the third head. Let X denote the number of tosses.

   (i) Construct the UMP (uniformly most powerful) test for the hypothesis.

   (ii) If you observe the third head occurs on the 12th toss, do you reject H_0 in a test of size $\alpha = 0.05$?

8. Let $X_1, \ldots, X_n$ be i.i.d. Poisson(λ), and let $\overline{X}_n, S_n^2$ denote the sample mean and variance, respectively.

   (i) Prove that $\overline{X}_n$ is the best unbiased estimator of λ (under the quadratic loss), without using the Cramer-Rao Theorem.

   (ii) Prove that

$$E(S_n^2 | \overline{X}_n) = \overline{X}_n$$

   and use it to show that

$$\mathrm{Var}(S_n^2) > \mathrm{Var}(\overline{X}_n).$$

   (iii) Using completeness, can a general theorem be formulated for which the identity in part (ii) is a special case?

9. Let $X_1, \cdots, X_n$ be a random sample with the p.d.f of the common distribution as

$$f(x|\theta) = \frac{(1+\theta x^2)}{\sqrt{2\pi}(1+\theta)} e^{-x^2/2}, \ -\infty < x < \infty, \ \theta > 0.$$

   (i) Find $E(X_1)$ and $\mathrm{Var}(X_1)$.

   (ii) Find the maximum likelihood estimate of θ.

   (iii) Find a "good" large sample testing statistic for testing $H_0 : \theta = 0$, v.s. $H_1 : \theta > 0$.