編號:______________

問題	1: 10分	2: 10分	3: 10分	4: 10分	5: 10分	總分: 100分
得分						
問題	6: 10分	7: 10分	8: 10分	9: 10分	10: 10分	
得分						

1. Let $M_X(t)$ be the moment generating function of X, and define $S(t) = \log(M_X(t))$. Show that
$$\frac{d}{dt}S(t)\bigg|_{t=0} = \mathrm{E}\,X \quad \text{and} \quad \frac{d^2}{dt^2}S(t)\bigg|_{t=0} = \mathrm{Var}\ X.$$

2. Let X and Y be independent $\mathrm{n}(0,1)$ random variables, and define a new random variable Z by
$$Z = \begin{cases} X & \text{if } XY > 0 \\ -X & \text{if } XY < 0. \end{cases}$$
   (a) Show that Z has a normal distribution.

   (b) Show that the joint distribution of Z and Y is not bivariate normal. (*Hint*: Show that Z and Y always have the same sign.)

3. Let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be the ordered values of n independent uniform $(0,1)$ random variables. Prove that for $1 \le k \le n+1$,
$$P\{X_{(k)} - X_{(k-1)} > t\} = (1-t)^n$$
where $X_{(0)} \equiv 0$, $X_{(n+1)} \equiv t$.

4. Let $X_1, \ldots, X_n$ be a random sample from a population with pdf
$$f_X(x) = \begin{cases} 1/\theta & 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$
Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent random variables.

5. Let $X_1, \ldots, X_n$ be iid $\mathrm{n}(\mu, \sigma^2)$. Find the best unbiased estimator of σ^p, where p is a known positive constant, not necessarily an integer.

6. Let X be an observation from the pdf

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1; \quad 0 \le \theta \le 1.$$

(a) Find the MLE of θ.

(b) Define the estimator $T(X)$ by

$$T(X) = \begin{cases} 2 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $T(X)$ is an unbiased estimator of θ.

7. Let X be one observation from a Cauchy(θ) distribution. Show that the test

$$\phi(x) = \begin{cases} 1 & \text{if } \quad 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

is most powerful of its size for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. Calculate the Type I and Type II Error probabilities.

8. Let $X_1, \ldots, X_n$ be iid observations from a beta($\theta, 1$) pdf and assume that θ has a gamma(r, λ) prior pdf. Find a $1 - \alpha$ Bayes credible set for θ.

9. Suppose that $X_1, \ldots, X_n$ are iid Poisson(λ). Find the best unbiased estimator of $e^{-\lambda}$, the probability that $X = 0$.

10. A random sample $X_1, \ldots, X_n$ is drawn from a population with pdf

$$f(x|\theta) = \frac{1}{2}(1 + \theta x), \quad -1 < x < 1, \quad -1 < \theta < 1.$$

Find a consistent estimator of θ and show that it is consistent.