Real and Complex Analysis
Ph.D. Qualifying Examination
Feb. 25, 2010

Do all the problems below in detail. Each problem carries 10%. Here m denotes the Lebesgue measure.

Part I: Real Analysis

(1) Let $\{g_n, n = 1, 2, \ldots\}$ be a sequence of uniformly bounded real functions on $\mathbf{R}$. Suppose that $f_n \to f$ in $\mathbf{L^p}$ and $g_n \to g$ a.e.. Show that $f_n g_n \to fg$ in $\mathbf{L^p}$.

(2) Let $[a, b]$ be a finite closed interval in $\mathbf{R}$. Show that all C^1 functions are of bounded variation over $[a, b]$.

(3) Let l^∞ be the space of bounded sequence of real numbers and define $\|\{a_n\}\|_\infty = \sup_n |a_n|$. Show that l^∞ equipped with the given norm is a Banach space.

(4) Let μ be a positive measure on a measurable space X. Suppose that $f \in L^1(X, \mu)$. Prove that for each $\epsilon > 0$, there is a $\delta > 0$ such that $\int_E |f| \, d\mu < \epsilon$ whenever $\mu(E) < \delta$.

(5) Let $E = \cup_{n=1}^\infty A_n$, $A_n \subset A_{n+1}$ for all $n \geq 1$. Let f be a nonnegative integrable function over E. Show that

$$\lim_{n\to\infty} \int_{A_n} f \, dm = \int_E f \, dm.$$

Part II: Complex Analysis

(6) Give a precise definition of a singled-valued branch of $\sqrt{z-2}\sqrt{z+2}$ in a suitable region such that it is analytic.

(7) Evaluate the improper integral

$$\int_{-\infty}^\infty \frac{dx}{1+x^4}.$$

(8) Suppose that f is holomorphic in $|z| < 1$. Is $\overline{f(\bar{z})}$ holomorphic in $|z| < 1$ also?

(9) Locate and classify the singularities of $f(z) = (1+\cos z)^{-1}$ in the complex plane.

(10) Find the image of $|z| = 4$ under the bilinear mapping $w = 1/(z+2)$.

End of Paper