Real and Complex Analysis

Ph.D. Qualifying Examination

September, 2009

Do all the problems below in detail. Each problem carries 10%. Here m denotes the Lebesgue measure.

(1) Are all real continuous functions on $\mathbf{R}$ Lebesgue measurable?

(2) Let g be integrable over $\mathbf{R}$ and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \le g$ and $f_n \to f$ on $\mathbf{R}$. Prove or disprove: $f_n \to f$ in L^1, i.e., $\int_{\mathbf{R}} |f_n - f| \, dm \to 0$.

(3) Suppose that f is a bounded nonnegative measurable function on $\mathbf{R}$. Let $\mu(A) = \int_A f(x)\, dm(x)$ for each measurable $A \subset \mathbf{R}$. Is μ absolutely continuous with respect to m?

(4) Let $(\mathbf{X}, \mathcal{F}, \mu)$ be a finite measure space. Give and justify the inclusion relationship between $L^1(\mu)$ and $L^2(\mu)$.

(5) State the Fubini theorem.

(6) Let $f(z) = \bar{z}^2/z$ for $z \neq 0$, and $f(0) = 0$. Find $f'(z)$ if it exists, and indicate where f is analytic.

(7) Let $p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$, where $z_1, z_2, \ldots, z_n$ are distinct complex numbers. Evaluate $\oint_C \frac{1}{p(z)}\, dz$, where C is the circle $|z| = R$ with $\max\{|z_1|, |z_2|, \ldots, |z_n|\} < R$. The integral is evaluated in the counterclockwise direction.

(8) Find the radius of convergence for $\sum_{n=1}^{\infty} n^2 3^n (z - i)^{n-1}$. Also, find the sum for those z inside the circle of convergence.

(9) Liouville's Theorem says that a bounded entire function is constant. Is it true that if an entire function has either a bounded real part or a bounded imaginary part, then it is constant?

(10) Let f be harmonic in a simply connected open region Ω in $\mathbf{C}$. Are all its derivatives harmonic therein also?