Real and Complex Analysis

Ph.D. Qualifying Examination

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Answer all the problems below. Each problem carries 10%. Here $m^*(A)$ denotes the outer (Lebesgue) measure of A.

(1) Let f be the function defined on the unit interval $(0,1)$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \end{cases}.$$

Show that f is not continuous at rational numbers but is continuous at irrational numbers in (0,1) .

(2) (a) Show that if $A \subset \mathbf{R}$ is measurable, then for any $\epsilon > 0$, there is some open set G and some closed set F such that $F \subset A \subset G$ and $m^*(G \setminus F) < \epsilon$.

(b) Define $m_*(A) = \sup\{\overline{m}(K) : K \text{ is a compact subset of } A\}$. Show that if A is measurable and $m(A) < \infty$, then $m_*(A) = m^*(A)$.

(3) (a) State the Fatou's lemma. Give an example of a sequence of functions (f_n) such that $f_n \to f$ a.e. on $(0,1)$, but $\underline{\lim}_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$.

(b) Let (g_n) be a sequence of integrable positive functions which converges a.e. to an integrable function g. Let (f_n) be a sequence of measurable functions which that $|f_n| \le g_n$ and f_n converges to f a.e. Show that if $\int g\, d\mu = \lim_{n\to\infty} \int g_n\, d\mu$, then

$$\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu.$$

(4) Let g be a positive, C^∞ function with compact support defined on $\mathbf{R}$, and $\int_{\mathbf{R}} g(x)\,dx = 1$. For any $f \in L^p(\mathbf{R})$ $(1 < p < \infty)$, define

$$f * g(x) := \int_{\mathbf{R}} f(x-y)g(y)\,dy.$$

(a) Show that $f * g = g * f$ on $\mathbf{R}$, whenever one of them exists.

(b) Show that $\|f * g\|_p \leq \|f\|_p$ for all $f \in L^p(\mathbf{R})$. (Hence $f * g \in L^p(\mathbf{R})$.)

Hint: Decompose $g = g^{1/p} + g^{1/q}$ and use Holder inequality.

(5) Let (X, M, μ) be a finite measure space and f be a measurable function on X. Set $a_n = \int_X |f|^n\,d\mu$.

(a) Show that $a_n^{1/n}$ converges to $\|f\|_\infty$.

(b) Show that $\frac{a_{n+1}}{a_n}$ also converges to $\|f\|_\infty$.

(6) Let μ, ν and λ be σ-finite measures. We say that $\nu << \mu$ when ν is absolutely continuous with respect to μ. Let the Radon-Nikodym derivative of ν with respect to μ be denoted by $\frac{d\nu}{d\mu}$.

(a) Show that if $\nu << \mu << \lambda$, then

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}.$$

(b) Show that if $\nu << \mu$ and $\mu << \nu$, then

$$\frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\nu}\right)^{-1}.$$

(7) Let Ω be an open connected set in $\mathbf{C}$, and f is analytic on Ω. Show that if $\{z \in \Omega : f(z) = 0\}$ has a limit point in Ω, then $f \equiv 0$.

Hint: Consider the Taylor series of f at the limit point.

(8) Let f be an entire function. If for all $z \in \mathbf{C}$, $|f(z)| \leq C|z|^n$, show that f is a polynomial of degree $\leq n$.

(9) Let a be a positive real number. Evaluate the improper integral

$$\int_0^{\infty} \frac{\sin x}{x(x^2+a^2)}\, dx \ .$$

(10) Let $\rho > 0$. Show that for n large enough, all the zeros of

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}$$

lie in the circle $|z| < \rho$.

End of Paper