PhD qualifying exam for analysis, Fall 2007

Part I: Real Analysis

1. Let X be a compact Hausdorff space and $C(X)$ be the vector space of real-valued continuous functions on X with norm

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}.$$

   (a) Show that $C(X)$ with the above norm is complete. (10pts)

   (b) A linear functional Λ on $C(X)$ is called positive if $\Lambda f \geq 0$ whenever $f \geq 0$. Show that if Λ is positive, then Λ must be bounded. (10pts)

2. Let (X, Ω, μ) be a measure space with $\mu(X) = 1$ and f be measurable on X.

   (a) Show that $\|f\|_p \leq \|f\|_q$ if $1 \leq p \leq q \leq \infty$. (10pts)

   (b) Show that either $\lim_{p\to\infty} \|f\|_p$ exists or $\|f\|_p$ diverges properly to ∞. (5pts)

   (c) Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ if $\|f\|_\infty < \infty$. (5pts)

   (d) Does the conclusion in (c) still hold if $\|f\|_\infty = \infty$? (5pts)

3. Show that there exists a Lebesgue measurable function f on $\mathbb{R}$ such that the set $\{x \in \mathbb{R} : f(x) \neq g(x)\}$ has positive Lebesgue measure for every $g \in C(\mathbb{R})$. (10pts)

4. Let Ω be a σ-algebra of subsets of X. Show that if Ω has infinitely many elements, then Ω is uncountable. Hint: Show that the relation "$\sim$" on X defined by $x \sim y$ if, for each $U \in \Omega$, either $\{x, y\} \subseteq U$ or $\{x, y\} \cap U = \emptyset$, is an equivalent relation on X. (15pts)

5. Let Γ be a set of real-valued functions on X. We say that $A, B \subseteq X$ are separated by Γ if there exist $a < b$ and $\varphi \in \Gamma$ such that $\varphi(x) < a$ for all $x \in A$ and $\varphi(y) > b$ for all $y \in B$. We say that an outer measure μ^* on X is a Carathéodory outer measure with respect to Γ if $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ whenever A, B are separated by Γ. It is known that if μ^* is an Carathéodory outer measure on X with respect to Γ, then every function in Γ is μ^*-measurable.

   (a) Let (X, ρ) be a metric space and μ^* be an outer measure on X such that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ whenever $\rho(A, B) > 0$. Show that every closed set in X is a μ^*-measurable set. (10pts)

(b) Let (X, ρ) be a metric space and $\alpha > 0$. For each $\epsilon > 0$, set

$$\lambda_\alpha^{(\epsilon)}(E) := \inf \sum_i r_i^\alpha, \quad E \subseteq X,$$

where $\{r_i\}$ are radii of a sequence of balls $\{B_i\}$ such that $r_i < \epsilon$ and $E \subseteq \bigcup B_i$. Show that $\lambda_\alpha^{(\epsilon)}$ increases as $\epsilon \to 0$. Defines

$$m_\alpha^*(E) := \lim_{\epsilon \to 0} \lambda_\alpha^{(\epsilon)}(E), \quad E \subseteq X.$$

Show that m_α^* is an outer measure and induces a Borel measure m_α on X. The measure m_α is called the Hausdorff α-dimensional measure on X. (10pts)

(c) Let E be a Borel set in X. Show that $m_\beta(E) = 0$ for all $\beta > \alpha$ if $m_\alpha(E) < \infty$ and $m_\beta(E) = \infty$ for all $0 < \beta < \alpha$ if $m_\alpha(E) > 0$. (10pts)

Part II: Complex Analysis

1. Let $p(z)$ be the polynomial given by

$$p(z) = \alpha(z - \alpha_1)^{k_1}(z - \alpha_2)^{k_2} \cdots (z - \alpha_n)^{k_n}, \quad \alpha \neq 0,$$

where $k_1 + k_2 + \cdots + k_n = \deg p(z)$. Show that if $\deg p(z) \geq 2$, then

$$\oint_C \frac{1}{p(z)} dz = 0$$

for any piecewise smooth closed curve in $\mathbb{C}$ if $\alpha_1, \alpha_2, \cdots, \alpha_n \in \text{inside}(C)$. (10pts)

2. An entire function f is periodic if there exists $z_0 \neq 0$ such that $f(z + z_0) = f(z)$ for all $z \in \mathbb{C}$. Show that if z_1 and z_2 are both periods of f, then f is constant if $z_1/z_2 \notin \mathbb{R}$. (10pts)

3. Show that if $f(z)$ is analytic on a domain D, then $f(z)$ must be constant if any one of the following condition is satisfied

(1) $f(z)$ is real-valued for all z in D; (5pts)

(2) $\overline{f(z)}$ is analytic on D; (5pts)

(3) $|f(z)|$ is constant on D. (5pts)

4. Let $u(x, y)$ be harmonic on $H = \{(x, y) : y > 0\}$, and is continuous and bounded upto the x-axis. Show that

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{u(t,0)dt}{(t-x)^2+y^2}. \quad \text{(10pts)}$$

5. (a) Prove the Casoratti-Weierstrass theorem: If z_0 is an essential singularity of an analytic function g, then $g(\{z : |z - z_0| < \varepsilon,\ z \neq z_0\})$ is dense for all $\varepsilon > 0$. (10pts)

   (b) Let f be an entire function on $\mathbb{C}$. Show that either f is constant or $f(\mathbb{C})$ is dense in $\mathbb{C}$. (15pts)

6. Let C be the positively oriented rectangle

$$\{(x,y) : -3 \leq x \leq 5,\ y = \pm 3\} \cup \{(x,y) : -3 \leq y \leq 3,\ x = -3, 5\}.$$

Evaluate

   (a) $\oint_C z^{3/2} dz$ (branch cut of $z^{3/2}$ is $\{x \leq 0\}$); (6pts)

   (b) $\oint_C \frac{dz}{z^2+16}$. (4pts)

7. Let $a \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Consider the function

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}.$$

   (a) Show that $\varphi_a(z)$ is analytic on $\mathbb{D}$ by computing $\varphi_a'(z)$. (5pts)

   (b) Show that $|\varphi_a(z)| = 1$ if $|z| = 1$. (5pts)

   (c) Show that $\varphi_a(z)$ sends $\mathbb{D}$ one-to-one and onto $\mathbb{D}$. (10pts)