Ph.D. Qualifying Examination
Real and Complex Analysis
8:30 – 12:00, 02/27/2007

10 points for each problem.

(1) Let m^* be the Lebesgue outer measure and E be a subset of $\mathbf{R}$. Show that E is Lebesgue measurable if and only if for each $\epsilon > 0$ there is an open set $O \supseteq E$ with $m^*(O \setminus E) < \epsilon$.

(2) Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. Show that $\int |f_n - f| \, d\mu \to 0$ if and only if $\int |f_n| \, d\mu \to \int |f| \, d\mu$.

(3) Let F be a nondecreasing right-continuous function on $[0, 1]$. Show that there are nondecreasing functions F_d and F_c such that $F = F_d + F_c$ where F_d is a step function with at most countably many jumps and F_c is continuous.

(4) Let $f_n \to f$ in L^p, $1 \le p < \infty$, and let $\{g_n\}$ be a sequence of measurable functions such that $|g_n| \le M < \infty$, for all n, and $g_n \to g$ a.e. Prove or disprove: $g_n f_n \to gf$ in L^p.

(5) Let μ and ν be signed measures such that ν is singular and absolutely continuous with respect to μ. Prove or disprove: $\nu = 0$.

(6) Evaluate

$$\int_0^\pi \frac{d\theta}{2 + \cos\theta}.$$

Hint: $z = e^{i\theta} = \cos\theta + i \sin\theta$.

(7) Give a precise definition of a single-valued branch of $(z^2 - 1)^{1/2}$ in a suitable region, and prove that it is analytic.

(8) Express $f(z) = \frac{1}{z^2 - z - 2}$ as a Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ in different region of z.

(9) Describe the image of the transformation

$$w = \frac{iz + e^{i\frac{\pi}{4}}}{z + e^{i\frac{\pi}{4}}},$$

where $z \in \mathbf{C}$.

(10) Let f be analytic in the whole complex plane and real on the real axis, purely imaginary on the imaginary axis. Prove or disprove: f is odd.