Qualified Examination: Real and Complex Analysis
February 2006

1. If $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ such that $0 \le f_n \le 1$ and such that $f_n(x) \to 0$ as $n \to \infty$ for every $x \in [0,1]$, show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

2. Suppose ϕ is a real function on R such that

$$\phi(\int_0^1 f(x)dx) \le \int_0^1 \phi(f)dx$$

for every real bounded measurable f. Prove that ϕ is convex.

3. Let X be a normed linear space and let X^* be its dual space with the norm

$$\|f\| = \sup\{|f(x)| : \|x\| \le 1\}.$$

   (a) Prove that X^* is a Banach space.

   (b) Prove that the mapping $f \to f(x)$ is, for each $x \in X$, a bounded linear functional on X^*, of norm $\|x\|$.

   (c) Prove that $\{\|x_n\|\}$ is bounded if $\{x_n\}$ is a sequence in X such that $\{f(x_n)\}$ is bounded for every $f \in X^*$.

4. Evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{3x^2+2}{(x^2+4)(x^2+9)}dx.$$

5. Let Ω be the upper half of the unit disc U. Find the conformal mapping f of Ω onto U that carries $\{-1,0,1\}$ to $\{-1,-i,1\}$. Find $z \in \Omega$ such that $f(z) = 0$.Find $f(i/2)$.