PH.D QUALIFYING EXAM : REAL ANALYSIS 2019 FALL

1.(10%) Assume $\{f_n\}$ is a sequence of measurable functions on $\mathbb{R}^n$. Does the inequality $\int_{\mathbb{R}^n}(\liminf f_n)dx \leq \liminf \int_{\mathbb{R}^n} f_n dx$ always hold ? Justify your result.

2.(15%) Given a measurable set $A \subset [0,1]$ with $|A| > 0$. Let $B = \cos(A) = \{cos(x), x \in A\}$. Show that the measure of B is strictly less than the measure of A, i.e $|B| < |A|$.

3.(15%) Let A, B be two measurable subsets of $\mathbb{R}^1$. Define $f(x) = |(A - x) \cap B|$. Evaluate $\int_{\mathbb{R}^1} f(x)$.

4.(15%) Let $E = [0, 20]$. Suppose $\{f_n(x)\}$ is a sequence of measurable functions that converge to a function $f(x)$ a.e. on E. Show that given $\epsilon > 0$ and $\delta > 0$, there exist a measurable set $A \subset E$ and a natural number K such that $|A| < \delta$ and $|f_n(x) - f(x)| < \epsilon$ whenever $n \geq K$ and $x \notin A$.

5.(15%) Is it possible to construct a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in L^p(\mathbb{R})$ for all $p \geq 1$ but $f \notin L^\infty(\mathbb{R})$? Show your result.

6.(15%) Suppose μ is a Borel measure on $\mathbb{R}^n$ and assume that there exists a constant $c > 0$ such that whenever a Borel set E satisfies $|E| = c$, then $\mu(E) = c$. Show that μ is absolutely continuous w.r.t Lebesgue measure.

7.(15%) Prove the generalized Hölder inequality, i.e. suppose $1 < p_i < \infty$ for each $1 \leq i \leq n$ and $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{r}$, show that $||f_1 \cdots f_n||_r \leq ||f_1||_{p_1} \cdots ||f_n||_{p_n}$.