National Sun Yat-sen University
2018 Real Analysis Ph.D. Qualifying Exam

1. (15 %) Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that for any $A, B \in \mathcal{A}$, we have the equality:
$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

2. (15 %) Let f be a function defined and bounded in $Q = \{(x,t) | 0 \le x \le 1,\ 0 \le t \le 1\}$. Suppose that
(1) $f(\cdot, t)$ is a measurable function of x for each t.
(2) the partial derivative $\frac{\partial f}{\partial t}(x,t)$ exists for each $(x,t) \in Q$
(3) $\frac{\partial f}{\partial t}(x,t)$ is bounded in Q.
Show that
$$\frac{d}{dt}\int_0^1 f(x,t)dx = \int_0^1 \frac{\partial f}{\partial t}(x,t)dx$$

3. (15 %) Let f be a non-negative real-valued Lebesgue measurable on $\mathbb{R}$. Show that if $\sum_{n=1}^{\infty} f(x+n)$ is integrable on $\mathbb{R}$, then $f = 0$ a.e. on $\mathbb{R}$.

4. (15 %) Let g be a non-negative integrable function and $\{f_n\}$ be a sequence of integrable functions such that $|f_n| \le g$ a.e. for all n. Show that if $f_n \to f$ in measure μ then $\lim_{n\to\infty} \int |f_n - f| d\mu = 0$.

5. (15 %) Let $(X, \mathcal{A}, \mu)$ be a measure space and f be an integrable function. Prove that for every $\epsilon > 0$ there is $E \in \Sigma$ such that $\mu(E) < +\infty$ and $\int_{X \setminus E} |f| < \epsilon$.

6. (a) (5 %) State Lebesgue Dominated Convergence Theorem
   (b) (10 %) Show that the Lebesgue Dominated Convergence Theorem holds if <u>a.e. convergence</u> is replaced by <u>convergence in measure</u>.

7. (10 %) Let f be a integrable function in $(-\infty, \infty)$. Evaluate
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x-n)\left(\frac{x}{1+|x|}\right) dx$$