Department of Applied Mathematics, National Sun Yat-sen University
Ph.D. Program Qualifying Exam
Real Analysis, Fall 2017

Answer all of the following questions. (20 points for each question)

1. Let $\{f_n\}$ be a sequence of (Lebesgue) measurable functions defined on a (Lebesgue) measurable set $X \subset \mathbb{R}$ with finite measure, i.e. $m(X) < \infty$. If $|f_n(x)| \le M_x < +\infty$ for all n and for all $x \in X$, show that given any $\epsilon > 0$, there is a closed set $F \subset X$ and a finite number M such that $m(X \backslash F) < \epsilon$ and $|f_n(x)| \le M$ for all n and all $x \in F$.

2. Suppose that $f : [0, 1] \to \mathbb{R}$ is Lebesgue integrable and satisfies
$$\int_0^1 f(x)x^n \, dx = 0$$
for all $n \in \mathbb{N} \cup \{0\}$. Show that $f = 0$ a.e. on $[0, 1]$.

3. Suppose that f is a positive function in $L^p((0, \infty), dx)$, where $p > 1$, and let
$$F(x) = \int_0^x f(t) \, dt, \qquad x \in [0, \infty).$$
Show that
$$F(x) \in o(x^{1/q})$$
as $x \to 0^+$ and $x \to +\infty$, where $1/p + 1/q = 1$.

4. If $f \in L^2(0, 2\pi)$, show that
$$\lim_{n\to\infty} \int_0^{2\pi} f(x) \cos(nx) \, dx = 0.$$

5. Let $p > 1$. Show that if $f_n \to f$ in L^p, then $f_n \to f$ in measure.

End of Paper