Real Analysis

Ph.D. Qualifying Examination(2016)
Department of Applied Mathematics, National Sun Yat-sen University

Answer all the problems below in detail.

1. (*20 points*) Let $f_n \to f$ in L^p, $1 \le p < \infty$ and $\{g_n\}$ be a sequence of bounded measurable functions and $g_n \to g$ a.e. Show that

$$f_n g_n \to fg \quad in \quad L^p.$$

2. (*20 points*) Let A be a continuous linear transform of the Banach space X onto the Banach space Y. Show that the image by A of the unit sphere in X contains a sphere about the origin in Y.

3. (*20 points*) Let x be an element in a normed vector space X. Show that there is a bounded linear functional f on X such that $f(x) = ||f||||x||$.

4. (*20 points*) Suppose $\{f_n\}$ is a sequence of measurable functions that converge to f a.e. on a bounded measurable set E. Show that given $\eta > 0$ there is a subset $A \subset E$ with $m(A) < \eta$ such that

$$f_n \to f \quad uniformly \quad on \quad E \setminus A.$$

5. (*20 points*) Let f be a real-valued function on $(-\infty, \infty)$. Show that f is continuous if and only if for each open set $O \subset \mathbb{R}$, $f^{-1}[O]$ is an open set.