Real Analysis

Ph.D. Qualifying Examination

Department of Applied Mathematics, National Sun Yat-sen University

February 21th, 2013

- Time: 1:00 p.m. $\sim$ 5:00 p.m.
- Answer all the problems below in detail. Here $m(A)$ denotes the Lebesgue measure of A.

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1. Prove or disprove that if $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, then the derivative f' of f is a Lebesgue measurable function. (7%)

2. Let φ be a convex function on $(-\infty, \infty)$ and f an integrable function on $[0,1]$. Show that
$$\int_{[0,1]} \varphi(f(t))dt \geq \varphi\left(\int_{[0,1]} f(t)dt\right). \qquad (8\%)$$

3. Let f be nonnegative and measurable on a measurable subset E of $\mathbb{R}$. Show that $\int_E f = 0$ *if and only if* $f = 0$ a.e. on E. (10%)

4. (a) State and prove the Fatou's Lemma. (10%)

   (b) Give an example to show that we may have strict inequality in Fatou's Lemma. (5%)

5. Let f be an integrable function on a measurable set X. Show that for each $\varepsilon > 0$, there is a $\delta > 0$ (depending on ε) such that $\left|\int_E f\right| < \varepsilon$ holds for any measurable subset E of X with $m(E) < \delta$. (15%)

6. Let f be a bounded measurable function on $[0,1]$. Show that $\lim\limits_{p\to\infty} \|f\|_p = \|f\|_\infty$. (10%)

7. Suppose that $\{f_n\}$ is a sequence of measurable functions that converge to a real-valued function f almost everywhere on a measurable set E of finite measure. Prove that for any given $\varepsilon > 0$, there is a closed subset F of E such that $m(E \backslash F) < \varepsilon$ and $\{f_n\}$ converges uniformly to f on F. (15%)

8. (a) Let $x, y \geq 0$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Show that $x^\alpha y^\beta \leq \alpha x + \beta y$. (10%)

   (b) State and prove the Hölder's inequality by using (a). (10%).

- End of Question Paper -