REAL ANALYSIS

Ph.D. Qualifying Examination
National Sun Yat-sen University
September 13, 2012

Do all the problems below in detail. Here m denotes the Lebesgue measure.

(1) Let $\{f_n\}$ be a sequence of real functions on $\mathbf{R}$ which converges to f in measure (Lebesgue measure). Does it converge to f in $L^1(\mathbf{R}, m)$? (10%)

(2) Let $\{f_n\}$ be a sequence of real functions in $L^p(\mathbf{R}, m)$, $1 \leq p < \infty$, which converges a.e. to $f \in L^p(\mathbf{R}, m)$.
(a) Show that if $f_n \to f$ in $L^p(\mathbf{R}, m)$ then $\|f_n\|_p \to \|f\|_p$. (5%)
(b) Show that if $\|f_n\|_p \to \|f\|_p$ then $f_n \to f$ in $L^p(\mathbf{R}, m)$. (10%)

(3) Let $C[-1, 1]$ be the space of real continuous functions which vanish at -1 and 1. Is $C[-1, 1]$ equipped with the L^1 norm a Banach space? (10%)

(4) Let $F(x) = \int_0^x f(t)\,dt$ for $x \in [0, 1]$, where f is an integrable function on $[0, 1]$. Is F of bounded variation over $[0, 1]$? (10%)

(5) Let $(\mathbf{X}, \mathcal{F}, \mu)$ be a measure space with $\mu(\mathbf{X}) < \infty$ and $1 \leq p < q < \infty$.
(a) Prove or disprove: $L^p(\mathbf{X}) \subseteq L^q(\mathbf{X})$. (8%)
(b) Prove or disprove: $\|f\|_p \leq \|f\|_q$. (7%)

(6) Let $(\mathbf{X}, \mathcal{F}, \mu)$ be a measure space and f a real measurable function on $\mathbf{X}$. Let $\mathcal{B}$ be the σ-algebra of Borel sets of $\mathbf{R}$. Set $\nu(E) = \mu(f^{-1}(E))$ for $E \in \mathcal{B}$. Show that ν is a measure on $\mathcal{B}$. (10%)

(7) Let $(\mathbf{X}, \mathcal{F}, \mu)$ be a measure space and A_n, $n = 1, 2 \ldots$, be measurable sets. Prove or disprove: $\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(\cap_{k=1}^{n} A_k)$. (10%)

(8) Let $(\mathbf{X}, \mathcal{F}, \mu)$ be a measure space with $\mu(\mathbf{X}) < \infty$. Set $F(f) = \int_{\mathbf{X}} f\,d\mu$ for $f \in L^2(\mathbf{X}, \mu)$.
(a) Show that F is a bounded linear functional on $L^2(\mathbf{X}, \mu)$. (10%)
(b) Find the norm of F. (10%)

End of Paper