NATIONAL SUN YAT-SEN UNIVERSITY
Department of Applied Mathematics
PhD Qualification Exam: Real Analysis, 16 February 2012

Marks: 100 Time : 4 Hours

QUESTION PAPER

This question paper consists of nine (9) questions. Attempt all of them. Each of Questions 1-7 is worth of 12 marks, and each of Questions 8-9 is worth of 8 marks.

Note: In the following, μ is the Lebesgue measure on $\mathbb{R}$, 'measure' means 'Lebesgue measure', 'measurable functions' mean Lebsgue measurable functions.

Questions

1. Let $\{a_n\}_{n=1}^{\infty}$ be a real sequence.

   1.1. Prove that if $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$, then $\{a_n\}$ converges.

   1.2. Find an example such that $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$, but $\{a_n\}$ fails to converge.

   1.3. If $\{a_n\}_{n=1}^{\infty}$ is nonnegative and satisfies the condition

   $$a_{n+1} \leq a_n + \delta_n \quad \text{for all } n \geq 1,$$

   where $\delta_n \geq 0$ for all n and such that $\sum_{n=1}^{\infty} \delta_n < \infty$, prove that $\lim_{n\to\infty} a_n$ exists.

2. Let f and f_n (for each integer $n \geq 1$) be measurable functions defined on a finite interval $[a, b]$.

   2.1. If $f_n \to f$ a.e. on $[a, b]$, prove that $f_n \to f$ in measure on $[a, b]$, that is,

   $$\lim_{n\to\infty} \mu(\{|f_n - f| \geq \delta\}) = 0 \qquad \text{for every fixed } \delta > 0.$$

   2.2. Prove that the set

   $$D = \{x \in [a, b] : \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

   is measurable.

   2.3. Suppose that $f_n \to f$ in measure and g is a continuous function on $\mathbb{R}$, prove that $g \circ f_n \to g \circ f$ in measure.

3. Let $E \subset \mathbb{R}$ be a measurable set with $\mu(E) > 0$ and let $f \in L(E)$. Let also $\{f_n\}$ be a sequence of measurable functions defined on E.

   3.1. If $f \geq 0$ and $\int_E f(x)dx = 0$, prove that $f(x) = 0$ a.e. on E.

   3.2. Prove that $\lim_{k\to\infty} k \cdot \mu(\{|f| \geq k\}) = 0$.

   3.3. If $\{f_n\}$ converges to f in measure, prove that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges to f on E almost everywhere.

4. Define a function f on [0,1] by setting $f(0) = 0$ and, for $x \in (0,1]$,

$$f(x) = \sum_{r_n < x} 2^{-n}$$

where $\{r_n\}_{n \geq 1}$ is the set of rational numbers in (0,1). Prove that f is strictly increasing on [0,1] and $f'(x) = 0$ a.e. .

5. Suppose that a real-valued function f is Lipschitz continuous on a finite interval $[a, b]$; that is, there exists a constant $M \geq 0$ such that

$$|f(x) - f(y)| \leq M|x - y| \quad \text{for all } x, y \in [a, b].$$

Prove that f is absolutely continuous and is of bounded variation on $[a, b]$.

6. Suppose $E \subset \mathbb{R}$ is measurable. Define a sequence of functions by

$$f_n(x) = n \int_0^{1/n} \chi_E(x + t)dt, \quad n = 1, 2, \cdots .$$

(Here χ_E is the indicator function of E; that is, $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$.) Prove the following:

6.1. each function f_n is absolutely continuous on every bounded interval;

6.2. $f_n \to \chi_E$ a.e. on $\mathbb{R}$;

6.3. for every bounded interval $[a, b]$, $\int_a^b |f_n(x) - \chi_E(x)|dx \to 0$.

7. Let $1 \leq p < \infty$. Let $f \in L^p(\mathbb{R})$.

7.1. Prove that

$$\lim_{\lambda \to \infty} \int_{|f| > \lambda} |f(x)|^p = 0 \quad \text{and} \quad \lim_{\lambda \to \infty} \int_{|x| > \lambda} |f(x)|^p = 0.$$

7.2. If $f \geq 0$ and if there exists a sequence of nonnegative functions $\{f_n\}$ in $L^p(\mathbb{R})$ such that

$$\liminf_{n \to \infty} f_n(x) \geq f(x) \quad \text{for } x \in \mathbb{R} \text{ and} \quad \lim_{n \to \infty} \|f_n\|_p = \|f\|_p,$$

prove that $\lim_{n \to \infty} \|f_n - f\|_p = 0$.

8. Let $f \in L^2[0,]$ and define $F(x) = \int_0^x f(t)dt$ for $x \in [0, 1]$. Prove that $\|F\|_2 \leq \frac{1}{\sqrt{2}}\|f\|_2$.

9. Suppose $\{f_n\}_{n=1}^{\infty} \subset L^2[0, 1]$ satisfies the conditions: (i) $f_n \to 0$ in measure and (ii) $\|f_n\|_2 \leq 1$ for all n. Prove that $\|f_n\|_1 \to 0$ as $n \to \infty$.

–End of Question Paper–