Real Analysis
Ph.D. Qualifying Examination
Feb. 25, 2010

Do all the problems below in detail. Each problem carries 10%. Here m denotes the Lebesgue measure.

(1) Let $\{g_n, n = 1, 2, \ldots\}$ be a sequence of uniformly bounded real functions on $\mathbf{R}$. Suppose that $f_n \to f$ in $\mathbf{L^p}$ and $g_n \to g$ a.e.. Show that $f_n g_n \to fg$ in $\mathbf{L^p}$.

(2) Let $[a, b]$ be a finite closed interval in $\mathbf{R}$. Show that all C^1 functions are of bounded variation over $[a, b]$.

(3) Let l^∞ be the space of bounded sequence of real numbers and define $\|\{a_n\}\|_\infty = \sup_n |a_n|$. Show that l^∞ equipped with the given norm is a Banach space.

(4) Let μ be a positive measure on a measurable space X. Suppose that $f \in L^1(X, \mu)$. Prove that for each $\epsilon > 0$, there is a $\delta > 0$ such that $\int_E |f|\, d\mu < \epsilon$ whenever $\mu(E) < \delta$.

(5) Let $E = \cup_{n=1}^\infty A_n$, $A_n \subset A_{n+1}$ for all $n \geq 1$. Let f be a nonnegative integrable function over E. Show that

$$\lim_{n\to\infty} \int_{A_n} f\, dm = \int_E f\, dm.$$

(6) Let m^* be the Lebesgue outer measure on $\mathbf{R}$ and E a measurable subset of $\mathbf{R}$. Show that for every $\epsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \backslash F) < \epsilon$.

(7) Let $(\mathbf{X}, \mathcal{F}, \mu)$ be a measure space and A_n, $n = 1, 2 \ldots$, measurable sets. Prove or disprove: $\mu(\cup_{n=1}^\infty A_n) = \lim_{n\to\infty} \mu(\cup_{k=1}^n A_k)$.

(8) Let $(\mathbf{X}, \mathcal{F}, \mu)$ be a measure space and g a nonnegative integrable function on $\mathbf{X}$. Set $\nu(E) = \int_E g\, d\mu$. Show that ν is a measure on $\mathcal{F}$.

(9) Let ν be as in (8) and f a nonnegative measurable function on $\mathbf{X}$. Show that $\int_E f\, d\nu = \int_E fg\, d\mu$ for $E \in \mathcal{F}$.

(10) Let μ be a finite Baire measure on the real line. Show that its cumulative distribution function F is a monotone increasing bounded function which is continuous on the right.

End of Paper