Real Analysis

Ph.D. Qualifying Examination

September, 2009

Do all the problems below in detail. Here m denotes the Lebesgue measure.

(1) Are all real continuous functions on $\mathbf{R}$ Lebesgue measurable? [15%]

(2) Let g be integrable over $\mathbf{R}$ and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ and $f_n \to f$ on $\mathbf{R}$. Prove or disprove: $f_n \to f$ in L^1, i.e., $\int_{\mathbf{R}} |f_n - f| \, dm \to 0$. [10%]

(3) Let f be a continuous function on $[0,1]$. Is f always of bounded variation over $[0,1]$? [10%]

(4) Let $(\mathbf{X}, \mathcal{F}, \mu)$ be a finite measure space. Give and justify the inclusion relationship between $L^p(\mu)$ and $L^q(\mu)$ for $1 \leq p < q < \infty$. [15%]

(5) Suppose that f is a nonnegative integrable function on $\mathbf{R}$. Let $\mu(A) = \int_A f(x) \, dm(x)$ for each measurable $A \subset \mathbf{R}$.

(a) Show that μ is a measure on $\mathbf{R}$. [10%]

(b) Is μ absolutely continuous with respect to m? [10%]

(c) Is m absolutely continuous with respect to μ? [10%]

(6) (a) State the Fubini theorem. [10%]

(b) Give an example showing that interchanging the order of integration is not allowable. [10%]