國立中山大學應用數學系博士班資格考：機率論

Qualifying Examination in **Probability Theory** for the Ph.D. Program

February, 2020

Note: The proofs and statements must be detailed.

1. (15%) Let X be exponential distributed random variables with mean $\lambda > 0$. Show that
$$\Pr(X > s + t \mid X > s) = \Pr(X > t), \quad s, t \geq 0.$$

2. (15%) Let X and Y be independent and identical random variables having uniform distribution over $[0, 1]$. Find the probability that $g(s) = s^2 + 2Xs + Y > 0$ for all $s \in \mathbb{R}$.

3. (20%) Let X be a non-negative random variable. Show that
$$E[X] < \infty \qquad \text{if and only if} \qquad \sum_{n=1}^{\infty} \Pr(X \geq n) < \infty.$$

4. (15%) Let $X_1, X_2, \ldots$ be independent and identical random variables with $E[X_1^4] < \infty$. Show that $(\sum_{i=1}^{n} X_i)/n \to E[X_1]$ almost surely.

5. For a probability space $(\Omega, \mathcal{F}, \Pr)$, let $\{\mathcal{F}_n\}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$. Assume that $E[X^2] < \infty$. Let $Z_n = E[X \mid \mathcal{F}_n]$, $n = 1, 2, 3, \ldots$.
(a)(10%) Show that $\{Z_n, \mathcal{F}_n\}$ is a martingale.
(b)(10%) Show that $E[Z_n^2] < \infty$ for all $n = 1, 2, 3, \ldots$.

6. (15%) Let $\{X_n\}$ be a Markov chain with state space $S = \{0, 1, 2, \ldots, a\}$, $a \geq 3$, and its transition probability
$$\Pr(X_{n+1} = j \mid X_n = i) = \begin{cases} 1 & \text{if} \quad i = j = 0 \text{ or } i = j = a, \\ p & \text{if} \quad j = i + 1, 1 \leq i \leq a - 1, \\ 1 - p & \text{if} \quad j = i - 1, 1 \leq i \leq a - 1, \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$
Let $T = \min\{n : X_n = 0\}$. Find $E[T \mid X_0 = 2]$.

End of Paper