國立中山大學應用數學系博士班資格考試：機率論 2013/09

(1) (a) Find the characteristic function of the density function $f(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$. (10%)

(b) Suppose $\psi(t) = 2/(3e^{it} - 1)$, $t \in \mathbb{R}$, is the characteristic function of X. Find the probability distribution of X. (10%)

(2) A fair coin is tossed repeatedly until three successive heads appear or three successive tails appear. Suppose that the first two toss results are head. Find the probability that three successive heads appear before three successive tails. (10%)

(3) Assume that $\{X_n\}_{n\geq 0}$ is a sequence of random variables with $E|X_n| < \infty$ for all n. Let $Y_n = X_n + bX_{n-1}$. If for each n,

$$E[X_{n+1} \mid X_0, X_1, \ldots, X_n] = aX_n - X_{n-1} \quad \text{almost surely,}$$

find real values a and b such that $\{Y_n\}_{n\geq 1}$ is a martingale with respect to $\{X_n\}_{n\geq 0}$. (10%)

(4) Let X and Y be independent and uniformly distributed over [0,1]. Compute the probability that $g(t) = t^2 + Xt + Y > 0$ for all $t \in \mathbb{R}$. (15%)

(5) Let X be a nonnegative random variable with $EX < \infty$. Show that $\lim_{n\to\infty} E[XI_{\{X>n\}}] = 0$, where $I_{\{\cdot\}}$ is the indicator function. (15%)

(6) Let $\{X_n\}_{n\geq 1}$ and X be random variables. Show that if $\sum_{n=1}^{\infty} \Pr(|X_n - X| \geq 1/n) < \infty$, then the sequence $\{X_n\}_{n\geq 1}$ convergence almost surely to X. (15%)

(7) Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables with $EX_n = 0$ and $E|X_n|^2 < \infty$. Show that if $\sum_{n=1}^{\infty} \frac{E[X_n^2]}{n^2} < \infty$, then $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k = 0$ almost surely. (15%)