國立中山大學應用數學所 暨 國立高雄大學統計學研究所
一佰學年度第二學期博士班資格考試

考試日期及時間：101 年 2月 16日 13：00 – 17：00 科目：機率論

1. (10%) Suppose that $A_1, A_2, \ldots$ are pairwise independent events and $\sum_{n=1}^{\infty} P(A_n) = \infty$. Show that $\sum_{i=1}^{n} 1_{A_i} / \sum_{i=1}^{n} P(A_i) \to 1$ in probability, where 1_{A_n} is the indicator function of A_n.

2. (10%) Suppose that $X_1, X_2, \ldots$ are independent exponential distributed random variables with mean 1. Let $M_n = \max\{|X_i| : 1 \leq i \leq n\}$. Show that $M_n/n \to 0$ almost surely.

3. (10%) Independently flip a fair coin 10 times. Find the probability that successive heads never appear.

4. (10%) Suppose the lifetimes of components in a renewal process with instant renewal are independently and exponentially distributed with constant rate 1. Find the probability that at least four components have been replaced by time t.

5. (20%) Let $X_1, X_2, \ldots$ be Poisson distributed random variables with $P(X_n = k) = n^k e^{-n}/k!$, $k = 0, 1, 2, \ldots$.
   (a) Find the characteristic function of X_n.
   (b) Show that the limiting distribution of $(X_n - n)/\sqrt{n}$ is the standard normal.

6. (20%) Suppose $X_1, X_2, \ldots$ are independent random variables with $P(X_n = n) = 1/n^3 = 1 - P(X_n = 1/n^2)$. Let $Y_n = \sum_{i=1}^{n} X_i$.
   (a) For all n, find $E[Y_{n+1}|Y_n]$.
   (b) Show that $\lim_{n\to\infty} Y_n$ exists almost surely.

7. (20%) On a southern Pacific island, suppose that the weather (either sunny or rainy) tomorrow will be the same as the weather today with probability 4/5.
   (a) Given that it is sunny on July 1, find the probability that it is sunny on July 4.
   (b) In a long run, what fraction of days are sunny.

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