國立中山大學應用數學系博士班資格考:機率論 2006/9

(1) The Cauchy density is $C_u(x) = \frac{1}{\pi}\frac{u}{u^2+x^2}, -\infty < x < \infty$, for $u > 0$.

(a) Show that $C_u * C_v = C_{u+v}$, where $(C_u * C_v)(y) = \int_{-\infty}^{\infty} C_v(y-x)C_u(x)dx$ is the convolution of C_u and C_v.(8pts)

(b) Show that if $X_1, X_2, \cdots, X_n$ are independent and have density C_u, then $(X_1 + X_2 + \cdots + X_n)/n$ has density C_u as well.(8pts)

(2) The triangular density is

$$f(x) = \begin{cases} 1-|x| & \text{if } x \in (-1,1) \\ 0 & \text{otherwise.} \end{cases}$$

Find the characteristic function $\phi(x)$ of the triangular distribution and show the following inversion formula holds,

$$\int_a^b f(x)dx = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\phi(t)dt.(16pts)$$

(3) Let $S_n = \sum_{i=1}^n X_i, n = 1, 2, \cdots$, where X_i's are iid random variables. Assume $\sup_{j\le N} P(|S_N - S_j| > \alpha) = c < 1$, show that

$$P(\sup_{j\le N}|S_j| > 2\alpha) \le \frac{1}{1-c}P(|S_N| > \alpha).(12pts)$$

(4) Assume X_j's are iid random variables with $E(X_n) = m,$, $E(X_n^2) = \sigma^2$ and $E(X_n^4) = \xi^4 < \infty$. Let $S_n = \sum_{i=1}^n X_i$, show that

$$P(\lim_{n\to\infty}\frac{S_n}{n} = m) = 1.(12pts)$$

(5) If the X_j's are uncorrelated and their second moments have a common bound, then $\frac{S_n - E(S_n)}{n} \to 0$ a.e., where $S_n = \sum_{i=1}^n X_i$.(13pts)

(6) Let $X_1, X_2, \cdots$ be iid random variables with the distribution function $F(\cdot)$ and let $M_n = Max[X_1, X_2, \cdots, X_n]$. Find the limiting distributions of

(a) $M_n - \alpha^{-1}\log n$, when $F(x) = 1 - e^{-\alpha x}, x > 0$.(8pts)

(b) $n^{\frac{1}{\alpha}} M_n$, when

$$F(x) = \begin{cases} 1-x^{-\alpha} & \text{if } x \ge 1 \\ 0 & \text{otherwise.(8pts)} \end{cases}$$

(7) Consider a random walk on the integers such that $P_{i,i+1} = p, P_{i,i-1} = q$, for all integer i $(0 < p < 1, p+q = 1)$.

(a) Show that

$$P_{0,0}^{2m} = \binom{2m}{m} p^m q^m, \text{ and } P_{0,0}^{2m+1} = 0.(5pts)$$

(b) Show the generating function of $u_n = P_{0,0}^n$, that is $P(x) = \sum_{n=0}^{\infty} u_n x^n$ equals $(1 - 4pqx^2)^{-1/2}$.(5pts)

(c) Show that the generating function of the recurrence time from state 0 to state 0 is $F(x) = 1 - \sqrt{(1 - 4pqx^2)}$.(5pts)