國立中山大學應用數學系博士班資格考:機率論 2006/2/9

(1) Let $A_1, A_2, \cdots$ be an independent sequence of events in a probability space $(\Omega, \mathcal{F}, P)$ and define the tail σ-field $\mathcal{T} = \cap_{n=1}^{\infty}\sigma(A_n, A_{n+1}, \cdots)$. Prove that for each event A in the tail σ-field $\mathcal{T}$, $P(A)$ is either 0 or 1.(12 %)

(2) Assume that $X_1, X_2, \cdots$ are iid random variables with $E(X_1) = \infty$. Prove $P(|X_n| \geq 0 \text{ i.o.}) = 1$.(12%)

(3) Suppose that A, B and C are positive, independent random variables with distribution functions F. Show that the quadratic $Az^2 + Bz + c$ has real zeros with probability(12%)

$$\int_0^\infty \int_0^\infty F(\frac{x^2}{4y})dF(x)dF(y).$$

(4) Suppose that $X_1, X_2, \cdots$ are identically distributed (not necessary independent). Show that $E[\max_{k\leq n}|X_k|] = o(n)$.(12%)

(5) Suppose that $\{X_n\}$ is an independent sequence and $E(X_n) = 0$. If $\sum_{n=1}^{\infty} Var(X_n) < \infty$, then $\sum_{n=1}^{\infty} X_n$ converges with probability 1.(12%)

(6) Find the characteristic functions of the following density functions,

(a) $f(x) = 1 - |x|, -1 < x < 1$;(6%)

(b) $f(x) = \frac{1-\cos x}{\pi x^2}, -\infty < x < \infty$.(7%)

(7) Let $X_1, X_2, \cdots$ be iid random variables with the distribution function $F(\cdot)$ and let $M_n = Max[X_1, X_2, \cdots, X_n]$. Find the limiting distributions of

(a) $M_n - \alpha^{-1}\log n$, when $F(x) = 1 - e^{-\alpha x}$;(7%)

(b) $n^{\frac{1}{\alpha}} M_n$, when $F(x) = 1 - x^{-\alpha}$, if $x \geq 1$; $= 0$, otherwise.(7%)

(8) Consider a discrete time Markov chain with states 0, 1 and the transition matrix

$$P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

(a) Prove for $n \geq 1$ (7%)

$$P^n = \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^n}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix}.$$

(b) Find the stationary distribution of the chain if it exists.(6%)