Qualifying Exam for Ph.D. Program of Numerical Linear Algebra, September, 10, 2009

1. Let
$$A = \begin{bmatrix} 375 & 374 \\ 752 & 750 \end{bmatrix}.$$

   (a) Compute A^{-1} and $\kappa_\infty(A)$, the condition number of A in ∞-norm. (5%)

   (b) Give b, δb, x and δx satisfying
$$Ax = b, \qquad A(x + \delta x) = b + \delta b$$
   so that $\frac{\|\delta b\|_\infty}{\|b\|_\infty} \simeq O(10^{-3})$, but $\frac{\|\delta x\|_\infty}{\|x\|_\infty} \simeq O(1)$. (7%)

   (c) Give b, δb, x and δx satisfying
$$Ax = b, \qquad A(x + \delta x) = b + \delta b$$
   so that $\frac{\|\delta x\|_\infty}{\|x\|_\infty} \simeq O(10^{-3})$, but $\frac{\|\delta b\|_\infty}{\|b\|_\infty} \simeq O(1)$. (8%)
   Hint: $A = \begin{bmatrix} a & a-1 \\ 2*(a+1) & 2*a \end{bmatrix}$, where $a = 375$, and consider the "*nearly*" null space and range of matrix A and A^{-1}, respectively.

2. (a) Verify that a triangular unitary matrix must be diagonal. (5%)

   (b) Show that any unitary matrix must be a product of Givens rotation matrices and Householder reflection matrices. (10%)
   Hint: Use the basic unitary matrices, Givens rotations or Householder reflections, to obtain the QR-factorization of the target unitary matrix.

3. Consider the least squares problem
$$\min_{x \in \mathbb{C}^n} \|b - Ax\|_2, \qquad (LS)$$
   where $A \in \mathbb{C}^{m \times n}$ and $b \in \mathbb{C}^m$ are given.

(a) Let $x \in \mathbb{C}^n$ be a solution of (LS) and define $r = b - Ax$. Verify that x solves the normal equation $A^H Ax = A^H b$ and the following enlarged system holds, (10%)

$$\begin{bmatrix} I & A \\ A^H & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

(b) Consider Wuzawa's iteration

$$x_{k+1} = x_k + \varpi Q^{-1} A^H r_k, \quad k = 0, 1, \ldots, \quad \text{and } x_0 \in \mathbb{C}^n \text{ is given,}$$

where $\varpi \in \mathbb{R}$ is the Wuzawa parameter, $r_k = b - Ax_k$ and $Q \approx A^H A$ is Hermitian and positive definite.

i.) Show that the eigenvalues of matrix $Q^{-1}A^H A$ are always real and nonnegative. (10%)

ii.) Determine an interval for the parameter ϖ such that Wuzawa's iteration converges. (10%)
Hint: using an upper bound of the eigenvalues of $Q^{-1}A^H A$.

iii.) Show that the iterative vectors x_k converges to a solution of problem (LS) whenever Wuzawa's iteration is convergent. (5%)

4. Show that for any square matrices A and $T^H = T$, (10%)

$$\|A - \frac{A + A^H}{2}\| \leq \|A - T\|.$$

5. Let A be an $m \times n$ matrix, having singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_{\min\{m,n\}}$. Verify that for any $m \times n$ matrix B with $\text{rank}(B) = k \leq r$,

$$\|A - B\|_2 \geq \sigma_{k+1}.$$

In addition, find a matrix B such that the equality can be achieved. (10%)

6. An $n \times n$ matrix A is strictly diagonally dominant if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad \text{for } i = 1, \ldots, n.$$

Here a_{ij}, $i, j = 1, \ldots, n$, denotes the (i, j)-th element of matrix A. Show that a strictly diagonally dominant matrix is nonsingular. (10%)