Qualifying Exam for the PH. D. Program of Numerical Linear Algebra.

Seven questions with the marks indicated.

1.(5) Give a definition of Mathematics.

2.(15) Suppose that there exists a root of $f(x) = 0$ and $0 < f'(x) \leq M$. Prove that $x_{n+1} = x_n - \lambda f(x_n)$ yields the convergence sequence x_n to the root for arbitrary $x_0 \in (-\infty, \infty)$ and $0 < \lambda < \frac{2}{M}$.

3.(20) Let $A \in R^{n \times n}, Cond.(A) = \{\frac{\lambda_{max}(A^T A)}{\lambda_{min}(A^T A)}\}^{\frac{1}{2}}$, where $\lambda_{max}(A)$ and $\lambda_{min}(A)$ are the maximal and the minimal eigenvalues of matrix A, respectively. Prove the following:
(1) $Cond.(AB) \leq Cond.(A)Cond.(B)$,
(2) $Cond.(UA) = Cond.(A)$, where $U \in R^{n \times n}$ is an orthogonal matrix.

4.(15) Consider the linear algebraic equations with their perturbations $Ax = b, A(x + \Delta x) = b + \Delta b$, where the matrix $A \in R^{n \times n}$ is symmetric and positive definite, and $x, \Delta x, b, \Delta b \in R^n$ are vectors. Prove

$$\frac{||\Delta x|}{||x||} \leq \frac{\lambda_n}{\lambda_1} \frac{||\Delta b||}{||b||},$$

where $||x||$ is the Euclidean norm of vector x, and λ_n and λ_1 are the maximal and the minimal eigenvalues of A, respectively.

5.(15) Determine the eigenvalues and the eigenvectors for both $a = 0$ and $a > 0$. Observe the behavior as $a \to 0$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & a \\ 0 & a & 1 \end{pmatrix}.$$

6.(15) Solve $Ax = b$ by the iteration method, $x^{n+1} = x^n - w(Ax^n - b)$, where w is parameter to be sought, and A is symmetric and positive definite with the eigenvalues λ_i to satisfy $0 < a \leq \lambda_i \leq b$. Find the optimal parameter w.

7.(15) To fix the function $y = f(x) = a\exp(bx)$ by the known data $(x_i, y_i), i = 1, 2, ..., n$, where $y_i > 0$ and $n \geq 2$. Design a least squares algorithm to find the optimal parameters a and b.