Numerical Differential Equation **Qualifying Exam** **Feb. 2010**

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

1. Let $\mathbb{R}(t, s)$ denote the set of (real) floating-point numbers on computers. Thus,

$$x \in \mathbb{R}(t, s) \text{ iff } x = f \cdot 2^e,$$

where $f = \pm(.\,b_{-1}b_{-2}\cdots b_{-t})_2$, $e = \pm(c_{s-1}c_{s-2}\cdots c_0\,.)_2$ and all b_i and c_j are binary digits.

(a) [**6 points**] What is the distance $d(x)$ of a positive normalized floating-point number $x \in \mathbb{R}(t, s)$ to its next larger floating-point number:

$$d(x) = \min_{\substack{y \in \mathbb{R}(t,s) \\ y > x}} (y - x)\,?$$

(b) [**7 points**] Determine the relative distance $r(x) = d(x)/x$, with x as in (a), and give upper and lower bounds for it.

(c) [**7 points**] Prove that

$$max_{x \in \mathbb{R}(t,s)}|x| = (1 - 2^{-t})2^{2^s - 1} \qquad min_{x \in \mathbb{R}(t,s)}|x| = 2^{-2^s}$$

2. Consider a quadrature formula of the type

$$\int_0^1 f(x)dx = -\alpha f'(0) + \beta f(\frac{1}{2}) + \alpha f'(1) + E(f).$$

(a) [**6 points**] Use the method of undetermined coefficients to find α, β such that the formula has maximum degree of exactness.

(b) [**7 points**] What is the precise degree of exactness of the formula obtained in (a)?

(c) [**7 points**] Use the Peano Kernel of the error functional E to express $E(f)$ in terms of the appropriate derivative of f reflecting the result of (b).

3. Consider the nonlinear equation $F(x) = 0$, where $F : \Omega \longmapsto \mathbb{R}^n, \Omega \subset \mathbb{R}^n$ is a C^1 function.

   (a) [**10 points**]Derive the Newton's method, namely for a given initial guess x_0 derive the formula for x_{k+1} in terms of x_k if Newton's method is used for approximate solution $F(x) = 0$.

   (b) [**10 points**]Assume that $F \in C^3$ and $F'(x_*)$ is non-singular, where x_* is a solution of $F(x) = 0$. Prove that the Newton's method is well defined if x_0 is sufficiently close to x_* and that the sequence of Newton iterates converges quadratically to the solution.

4. [**20 points**] Consider the boundary value problem

$$-u'' + u' + u = f \quad \text{on } [0,1], \quad u'(0) = u'(1) = 0.$$

   Take

$$V = H^1(0,1)$$

$$a(u,v) = \int_0^1 (u'v' + u'v + uv)\, dx$$

$$F(v) = (f, v)$$

   Prove that $a(\cdot,\cdot)$ is continuous and V-elliptic(coercive). If the above differential equation is changed to

$$-u'' + ku' + u = f.$$

   Show that $a(\cdot,\cdot)$ need not be coercive for large k.

5. [**20 points**] Consider the Possion's equation in polar coordinates,

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = f(r,\theta),$$

   with $0 \le r \le s(\theta)$ and $0 \le \theta \le 2\pi$. Change coordinates to (ρ, ϕ) given by $\rho = \frac{r}{s(\theta)}$ and $\phi = \theta$. Derive (and prove) a finite difference scheme in a form that gives a positive definite and symmetric matrix.