Qualified Exam for the Ph.D of Numerical DEs

September 2009

Five questions with the marks indicated. Please write down all the detail of your computation and answers.

1.(15%) Let $f(x) \in C^4[a,b]$, and let $s(x)$ be the piecewise cubic Hermite interpolant of $f(x)$ relative to the partition

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Show that

$$\max_{x\in[a,b]} |f(x) - s(x)| \le \frac{h^4}{384} \max_{x\in[a,b]} |f^{(4)}(x)|,$$

where $h = \max_{0\le i\le n-1} (x_{i+1} - x_i)$.

2.(10%+10%) Let $f(x) \in C^4(D)$ and $\max_{x\in D} |f^{(4)}(x)| = M$.

(a) Derive an $O(h^2)$ three-point formula to approximate $f''(x_1)$ that uses $f(x_0)$, $f(x_1)$ and $f(x_2)$, where $x_i = x_0 + ih \in D,\ i = 0,1,2.$

(b) Assume the actual values used is $\tilde{f}_i$ with

$$f(x_i) = \tilde{f}_i + \varepsilon_i, \quad |\varepsilon_i| \le E, \qquad i = 0,1,2.$$

Derive the errors bounded of the actual numerical derivative computed and find an optimal value of $h > 0$.

3.(10%+10%) Consider the boundary value problem

$$-\Delta u = f, \quad (x,y) \in \Omega\ ; \qquad u_\nu = g, \quad (x,y) \in \partial\Omega = \Gamma. \tag{1}$$

(a) Variational formulation: Find $u \in V$ such that $a(u,v) = L(v)$, $\forall v \in V$. What are the appropriate choices of a Hilbert space V, a bilinear form $a(\cdot,\cdot)$ and a linear functional $L(\cdot)$ for the variational formulation of (1)?

(b) Show that a solution exists only if

$$\iint_\Omega f\ dxdy + \int_\Gamma g\ ds = 0,$$

and that if u is a solution then so is $u + c$, where c is an arbitrary constant.

4.(10%+10%) Consider the initial value problem

$$y'(x) = f(x, y(x)), \quad x \in [a, b]; \qquad y(a) = \alpha. \tag{2}$$

(a) Derive Euler's method with the stepsize h to solve (2) with local truncation error.

(b) Show that Euler's method fails to approximate the solution $Y(x) = (\frac{2}{3}x)^{\frac{3}{2}}$, $x \geq 0$, of the problem $y' = y^{\frac{1}{3}}$, $y(0) = 0$. Explain why.

5.(7%+10%+8%) Consider the heat equations

$$\begin{array}{l} u_t = u_{xx}, \quad t \geq 0, \quad 0 \leq x \leq 1, \\ u(0, t) = u(1, t) = 0, \quad t \geq 0, \\ u(x, 0) = f(x), \quad x \in [0, 1], \end{array} \tag{3}$$

where $f(x)$ is continuous.

(a) Construct an explicit scheme in solving (3) numerically.

(b) Discuss the convergent and stability properties of your scheme.

(c) Construct an implicit scheme to improve the stability of (a).