Ph.D. Qualifying Examination

Matrix Theory

Feb. 13, 2014

Please write down all the details of your computation and answers.

[1]. (15%) Let $A = \begin{bmatrix} 4 & 1 & -1 \\ -2 & 1 & 2 \\ -2 & -1 & 3 \end{bmatrix}$. Compute the limit of the series $\sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}A\right)^n}{n!}$.

[2]. (15%) Let $\mathcal{S}$ be the hyperplane $x - y + z - w = 0$ in $\mathbb{R}^4$. Find an orthogonal basis on $\mathcal{S}$ and the projection matrix onto $\mathcal{S}$.

[3]. (15%) Let A be an $m \times n$ real matrix and B be an $n \times m$ real matrix. Prove or disprove that for a nonzero eigenvalue λ of AB, λ is also an eigenvalue of BA.

[4]. (15%) Let A be similar to B. Prove or disprove that $\|A\|_2 = \|B\|_2$ and $\|A\|_F = \|B\|_F$, where $\|\cdot\|_2$ is the induced matrix 2-norm and $\|\cdot\|_F$ is Frobenius norm.

[5]. (10%) Show that the spectral radius $\rho(A) = \lim_{k\to\infty} \left\|A^k\right\|^{\frac{1}{k}}$ for any matrix norm $\|\cdot\|$.

[6]. (15%) State and prove the Cayley-Hamilton theorem for matrices with complex entries.

[7]. (15%) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}$ and $A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \in \mathbb{R}^{4\times 3}$. The singular values of A are arranged in the order $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and the singular values of A' are arranged in the order $\sigma_1' \geq \sigma_2' \geq \sigma_3'$. Show that $\sigma_1' \geq \sigma_1 \geq \sigma_2' \geq \sigma_2 \geq \sigma_3' \geq \sigma_3$.

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