Ph.D. Qualifying Examination

Matrix Theory

September 14, 2012

Please write down all the detail of your computation and answers.

[1]. (15%) Let $A = (a_{ij})_{n\times n}$ be a matrix over complex field, $\|A\|_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}$ be its Frobenius norm, and $\|A\|_2$ be the induced matrix 2-norm compatible with Euclidean norm. Show that for any unitary matrices U and V, (1) $\|UAV\|_F = \|A\|_F$, (2) $\|UAV\|_2 = \|A\|_2$, and (3) $\|A\|_2 \le \|A\|_F \le \sqrt{n}\,\|A\|_2$.

[2]. (10%) State the Courant-Fischer theorem which characterizes the k-th eigenvalue of a Hermitian matrix.

[3]. (12%) Recall that a matrix is nonnegative if all its entries are nonnegative. Describe the Perron-Frobenius Theorem for nonnegative matrices.

[4]. (12%) (a) State the Singular Value Decomposition (SVD) Theorem for $A \in \mathbb{C}^{m\times n}$.

(b) Suppose that a matrix $A \in \mathbb{C}^{m\times n}$ has rank r. Construct r rank-one matrices H_i $(i = 1, 2, \cdots, r)$ so that $A = \sum_{i=1}^{r} H_i$.

[5]. (12%) State and prove Sylvester's law of inertia..

[6]. (12%) Prove that a complex square matrix is unitarily diagonalizable if and only if it is normal.

[7]. (15%) Let

$$A = \begin{bmatrix} 1 & 2 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Find the similarity transformation to convert matrix A to its Jordan canonical form.

[8]. (12%) Compute the QR factorization of

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

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