Ph.D. Qualifying Examination
Matrix Theory
Feb. 12, 2009

Please write down all the detail of your computation and answers.

(1) (15%) Let A be an $n \times n$ real matrix. Suppose that $A^3 = A$. For each $t \in (-1, 1)$, is $I + tA$ invertible? If yes, find its inverse.

(2) (20%) Let

$$A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}.$$

Find a matrix P such that $P^{-1}AP$ is in Jordan canonical form.

(3) (20%) Let A and B be $n \times n$ real matrices.
(i) Prove or disprove: AB and BA have the same eigenvalues, counting multiplicities.
(ii) Prove or disprove: AB and BA are similar.

(4) (20%) State and prove the Rayleigh quotient principle (Rayleigh-Ritz Theorem) for a Hermitian matrix.

(5) (15%) An $n \times n$ complex matrix A is said to be orthogonal if $AA^T = I$. Prove or disprove: If A is unitary, then A is orthogonal.

(6) (10%) Let A be an $n \times n$ complex matrix such that $< \mathbf{x}, A\mathbf{x} >= 0$ for all $\mathbf{x} \in \mathbf{C}^n$. Prove or disprove: A is the zero matrix.