Ph.D. Qualifying Examination
Matrix Theory
Sep. 11, 2008

Please write down all the detail of your computation and answers.

(1) (15%) Let A, B be $n \times n$ real matrices. Suppose that A and B are diagonalizable and $AB = BA$. Prove or disprove: A and B can be simultaneously diagonalizable, i.e., there is a P such that $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.

(2) (20%) Find e^{At}, $t \in \mathbf{R}$, where $e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$ and

$$A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}.$$

(3) (15%) Find a spectral decomposition, i.e., $A = \sum \lambda_i P_i$, where P_i is the projection onto the eigenspace for the eigenvalue λ_i, for

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(4) (15%) The *spectral radius* $\rho(A)$ of a complex square matrix A is defined as $\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$. Prove or disprove: $\rho(A) \leq \|A\|$ for any matrix norm $\|\cdot\|$.

(5) (20%) Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

(a) Show that it is possible to decompose A into the product of a positive definite matrix P and an orthogonal matrix U.

(b) Find P and U in (a).

(6) (15%) Assume that A is an invertible real matrix, $\mathbf{y} \neq 0$, and $A\mathbf{x} = \mathbf{y}$. For a given $\delta\mathbf{y}$, let $\delta\mathbf{x}$ be the vector that satisfies $A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{y} + \delta\mathbf{y}$. Let $\text{cond}(A) = \|A\| \cdot \|A^{-1}\|$ where $\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$. Show that

(a) $$\frac{1}{\text{cond}(A)} \frac{\|\delta\mathbf{y}\|}{\|\mathbf{y}\|} \leq \frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \text{cond}(A) \frac{\|\delta\mathbf{y}\|}{\|\mathbf{y}\|} \quad \text{(for any norm} \|\cdot\|\text{)}.$$

(b) $\text{cond}(A) = \sqrt{\lambda_{max}/\lambda_{min}}$ where λ_{max} and λ_{min} are the largest and smallest eigenvalues, respectively, of $A^T A$.