Quality Exam for Matrix Theory

Questions with the marks indicated.

February 26, 2007

1. (5) Describe the Courant-Fische theorem, i.e., the theorem of variational characterizations for eigenvalues of Hermitian matrices.

2. (10) Describe the singular value decomposition for a real matrix.

3. (10) Let $X, Y \in R^n$, and $Y^T X \neq -1$. Show $B^{-1} = I - \frac{1}{1+Y^TX} XY^T$, where the matrix $B = I + XY^T$ and I is the unity matrix.

4. (15) Let $A \in R^{m \times n}$. Prove $\frac{1}{2}(||A||_1 + ||A||_\infty) \leq \sqrt{n}||A||_F$, where $||A||_1 = max_j \sum_{i=1}^m |a_{ij}|$, $||A||_\infty = max_i \sum_{j=1}^n |a_{ij}|$, and $||A||_F = \sqrt{\sum_{i,j} a_{ij}^2}$.

5. (15) Let $A \in C^{n \times n}$, $Y \in C^n$, $a \in C$, and the Hermitian matrix

$$B = \begin{pmatrix} A & Y \\ Y^* & a \end{pmatrix} \tag{1}$$

Prove the condition numbers have the bounds: $Con_2.(A) \leq Con_2.(B)$, where $Con_2.(A) = ||A||_2||A^{-1}||_2$.

6. (15) Determine the eigenvalues and eigenvectors for both $a = 0$ and $a > 0$. Observe the behavior as $a \to 0$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & a \\ 0 & a & 1 \end{pmatrix} \tag{2}$$

7.(15) Let $Ax = b$, $A\tilde{x} = \tilde{b}$, where $|A| \neq 0$. Prove

$$\frac{||x - \tilde{x}||}{||x||} \leq ||A|| ||A^{-1}|| \frac{||b - \tilde{b}||}{||b||}$$

where $x \neq 0$, $b \neq 0$ and $||x||$ is any vector norm.

8.(15) Solve the over-determined system

$$Ax = b, \; A \in R^{m \times n}, \; x \in R^n, b \in R^m, \; m > n,$$

with $Rank(A) = n$. Give a solution method and the corresponding stability analysis.

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