Ph.D. Qualifying Examination
Matrix Theory
Sep. 14, 2006

Please write down all the detail of your computation and answers.

1. (10%) Find the third column of the following product matrix

$$\begin{bmatrix} \pi & \sqrt{e} & \frac{1}{3} & \sqrt{2} \\ 3.7 & 10^5 & 7 & 0 \\ \ln 2 & i & \sin 3 & -1 \end{bmatrix} \begin{bmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ 0.2 & -0.3 & 0.1 \\ 2 & -1 & -1 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} -1 & 2.3 & 1 & -3 \\ \sqrt{5} & \frac{1}{2} & 1 & 2 \\ 3 & -2 & 1 & \sqrt{2} \end{bmatrix}.$$

2. (15%) Let A be an diagonalizable matrix and t be a parameter. Consider the linear equation $(A - tI)\mathbf{x} = \mathbf{b}$. (1) Discuss the existence and uniqueness of $\mathbf{x}$ in terms of the values of t and $\mathbf{b}$, eigenvalues and eigenvectors of A. (2) Find all of its solutions if they exist.

3. (15%) Let A and B be two real matrices. Without considering the multiplicity, show that AB and BA have the same eigenvalues.

4. (15%) (1) State the Fundamental Theorem of Linear Algebra. (2) Use it to prove the existence of singular value decomposition of any $m \times n$ complex matrix.

5. (15%) What is the relation between eigenvalues and singular values of a square matrix A if A is (1) normal, (2) Hermitian, (3) Hermitian positive definite? State the reasons.

6. (15%) (1) State all the equivalent conditions you know for a matrix to be positive definite. (2) Prove all your conditions are equivalent.

7. (15%) (1) Use the Geršgorin Disk Theorem to prove that a strictly diagonally dominant matrix is nonsingular. (2) Use the nonsingularity of strictly diagonally dominat matrix to prove the Geršgorin Disk Theorem.