# Ph.D. Qualifying Examination
## Matrix Theory
## Feb. 10, 2006

*Please write down all the detail of your computation and answers.*

1. (15%) For every $n \times n$ complex matrix $A = (a_{ij})_{n \times n}$, define

$$||A|| = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}.$$

Show that it is a matrix norm compatible with the vector norm

$$||(x_1, x_2, \cdots, x_n)^T|| = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

2. (15%) (1) State the singular value decomposition for $m \times n$ complex matrix.

   (2) Show that such decompostion exists for all $m \times n$ complex matrices.

3. (15%) Let $A$ be an $m \times n$ complex matrix with rank $r$. Find the necessary and sufficient conditions on $m$, $n$ and $r$ such that the number of solutions of $A\mathbf{x} = \mathbf{b}$ is (1) 0 or 1 depending on $\mathbf{b}$; (2) $\infty$ for every $\mathbf{b}$; (3) 0 or $\infty$ depending on $\mathbf{b}$; (4) 1 for every $\mathbf{b}$. Prove your answers.

4. (15%) Let $A$ be an $n \times n$ complex matrix. Show that the followings are equivalent: (1) $A$ is Hermitian, (2) $\mathbf{x}^* A\mathbf{x} \in \mathbf{R}$ for all $\mathbf{x} \in \mathbf{C}^n$, (3) $A$ is normal with all eigenvalues real, (4) $S^*AS$ is Hermitian for all $n \times n$ complex matrix $S$.

5. (15%) Let

$$A = \begin{pmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{pmatrix}.$$

   (1) For $\mathbf{x} \in \mathbf{R}^4$, write the quadratic form $\mathbf{x}^T A\mathbf{x}$ as a linear combination of some squares.

   (2) Find and prove the necessary and sufficient condition on $a$ and $b$ such that $A$ is symmetric positive definite.

6. (15%) Let $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ be two real matrices.

   (1) If $0 \leq a_{ij} \leq b_{ij}$ for all $i$ and $j$, show that the spectral radiuses $\rho(A) \leq \rho(B)$.

   (2) If $0 \leq a_{ij} < b_{ij}$ for all $i$ and $j$, show that $\rho(A) < \rho(B)$.

   (3) Is it possible that $0 \leq a_{ij} \leq b_{ij}$ for all $i$ and $j$, and $A \neq B$, but $\rho(A) = \rho(B)$? Why?

7. (10%) For every complex square matrix $A$, show that there are a diagonalizable matrix $D$ and a nilpotnet matrix $N$ such that $A = D + N$ and $DN = ND$.