Qualified Examination–Mathematical Programming

February 2014

20 points for each of the following problems

1. Let A be an $m \times n$ matrix. Show that exactly one of the following two systems has a solution.
   System 1 $\quad Ax > 0$
   System 2 $\quad A^t y = 0, y \geq 0, y \neq 0$
2. Consider the function θ defined by the following optimization problem: $\theta(u_1, u_2) =$ Minimum $x_1(1-u_1) + x_2(1-u_2)$ subject to $x_1^2 + x_2^2 \leq 1$.
   a. Show that θ is concave.
   b. Evaluate θ at the point $(1, 1)$.
   c. Find the collection of subgradients of θ at $(1, 1)$.
3. Solve the following problem: Miniimize $x_1 + 3x_2 + x_3$ subject to $x_1 + 4x_2 + 3x_3 \leq 12, -x_1 + 2x_2 - x_3 \leq 4, x_1, x_2, x_3 \geq 0$
4. Let $f : R^n \to R$, $g_i : R^n \to R$ be convex functions, $i = 1, 2, ..., m$. Consider the problem to minimize $f(x)$ subject to $g_i(x) \leq 0$ for $i = 1, 2, ..., m$. Let M be a proper subset of $\{1, ..., m\}$ and suppose that $\hat{x}$ solves the problem to minimize $f(x)$ subject to $g_i(x) \leq 0$ for $i \in M$. Let $V = \{i : g_i(\hat{x}) > 0\}$. If $\bar{x}$ solves the original problem, show that $g_i(\bar{x}) = 0$ for some $i \in V$.
5. Suppose that $f : R^n \to R$ is twice differentiable at x. If $\nabla f(x) = 0$ and $H(x)$ is positive definite, then x is a local minimum, where $H(x)$ is the Hessian matrix of f at x.