Qualified Examination: Mathematical Programming
September 2006

1. Consider the problem to minimize $f(x)$ subject to $Ax \leq b$. Suppose that x is a feasible solution such that $A_1 x = b_1$ and $A_2 x < b_2$ where $A^t = (A_1^t, A_2^t)$ and $b^t = (b_1^t, b_2^t)$. Assume that A_1 has full rank, the matrix P that projects any vector in the null space of A_1^t is given by $P = I - A_1^t(A_1 A_1^t)^{-1} A_1$.

   a. Let $d = -P\nabla f(x)$. Show that if $d \neq 0$, then it is an improving feasible direction;

   b. Suppose that $d = 0$ and that $u = -(A_1 A_1^t)^{-1} A_1 \nabla f(x) \geq 0$. Show that x is a Kuhn-Tucker point;

   c. Show that d generated above is of the form λv for some $\lambda > 0$ where v is an optimal solution of the following problem: Minimize $\nabla f(x)^t w$ subject to $A_1 w = 0$ and $\|w\|^2 \leq 1$;

2. Consider the function θ defined by the following optimization problem: $\theta(u, v)$=Minimize $x(1-u) + y(1-v)$ subject to $x^2 + y^2 \leq 1$.

   a. Show that θ is concave;

   b. Evaluate $\theta(1, 1)$;

   c. Find the collection of subgradients of θ at $(1, 1)$.

3. Let A be a $p \times n$ matrix and B be a $q \times n$ matrix. Show that exactly one of the following systems has a solution.

   System 1 $\quad Ax < 0 \quad Bx = 0 \quad$ for some $x \in R^n$

   System 2 $\quad A^t u + B^t v = 0 \quad$ for some (u, v), $u \neq 0$, $u \geq 0$.

4. Let K be a closed convex subset in R^n and $f : K \to R^n$ be a differentiable and convex function. Show that $x \in K$ is a solution to the problem: Minimize $f(y)$ subject to $y \in K$ if and only if x is a solution of the following problem: Find $y \in K$ such that $\langle \nabla f(y), v - y \rangle \geq 0$ for all $v \in K$.

5. Let $f : R^n \to R$ be convex. Show that ξ is a subgradient of f at x if and only if the hyperplane $\{(x, y) : y = f(\bar{x}) + \xi^t(x - \bar{x})\}$ supports epi$f$ at $[\bar{x}, f(\bar{x})]$.