Qualified Examination: Mathematical Programming
February 2006

1. Let $f : S \rightarrow R$ be a convex and differentiable function where S is a closed and convex subset in R^n. Show that $x \in S$ is a solution of the problem $\text{Min}_{y \in S} f(y)$ if and only if x is a solution of the following problem:

$$\langle \nabla f(x), y - x \rangle \quad \forall y \in S.$$

2. Let f be twice continuously differentiable on the open convex set $C \subset R^n$. Show that f is convex on C if and only if its Hessian matrix $\nabla^2 f(x)$ is positive semi-definite for all $x \in C$.

3. Solve the following problem:

$$\begin{array}{rl} \text{Minimize} & -x_1 - 2x_2 + x_3 \\ \text{Subject to} & x_1 + x_2 + x_3 \leq 4 \\ & -x_1 + 2x_2 - 2x_3 \leq 6 \\ & 2x_1 + x_2 \leq 5 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

4. Consider the problem: Minimize $x_1 + 2x_2$ subject to $3x_1 + x_2 \geq 6$, $-x_1 + x_2 \leq 2$, $x_1 + x_2 \leq 8$ and $x_1, x_2 \geq 0$. Let $X = \{(x_1, x_2) : -x_1 + x_2 \leq 2, x_1 + x_2 \leq 8, x_1, x_2 \geq 0\}$.

   a. Formulate the Lagrangian dual problem.

   b. Show that $f(w) = 6w + \text{Minimum}\{0, 4 - 2w, 13 - 14w, 8 - 24w\}$.

   c. Plot $f(w)$ for each value of w.

   d. From part (c) locate the optimal solution to the Lagrangian dual problem.

   e. From part (d) find the optimal solution to the primal problem.

5. Let X be a metric space and $f : X \rightarrow R \cup \{\infty\}$. Prove that f is lower semicontinuous if and only if epi$f$ is closed.