Department of Applied Mathematics, National Sun Yat-sen University
Ph.D. Program Qualifying Exam
Functional Analysis, Spring 2019

Answer any 5 of the following questions (20 points for each question).

In the following, $\mathcal{H}$ always represents a complex Hilbert space and $B(\mathcal{H})$ stands for the set of bounded linear operators on $\mathcal{H}$. Besides, $\mathcal{A}$ means a Banach algebra with identity e.

1. Let φ be a semi-inner product on a linear space X and $N = \{x \in X : \varphi(x, x) = 0\}$.
   (1) Show that N is a linear subspace of X.
   (2) Show that $\langle x + N, y + N\rangle = \varphi(x, y)$ for $x + N$ and $y + N$ in the quotient space X/N induces an inner product on X/N.
2. Suppose that $\mathcal{E} = \{e_n\}$ is an orthonormal basis of $\mathcal{H}$ and $T : \mathcal{E} \to \mathcal{H}$ is a map so that $\sum \|Te_n\| < \infty$. Prove that T extends uniquely as an element in $B(\mathcal{H})$.
3. Let P and Q be orthogonal projections of $\mathcal{H}$ onto two closed subspaces of $\mathcal{H}$. Show that $P + Q$ is an orthogonal projection if and only if $P \perp Q$.
4. Let $T \in B(\mathcal{H})$ be a self-adjoint operator with $\|T\| \leq 1$. Prove that $T = 2S(1 + S^2)^{-1}$ for some self-adjoint operator $S \in B(\mathcal{H})$.
5. Let $T \in B(\mathcal{H})$ be a normal operator. Show that $T^* = T$ if and only if $\exp(iT)$ is a unitary operator.
6. Let μ be a compactly supported probability measure on the real line. Show that there exist a Hilbert space $\mathcal{H}$, a positive linear functional ϕ on $B(\mathcal{H})$, and an operator $T \in B(\mathcal{H})$ for which
$$\phi(f(T)) = \int f \, d\mu$$
holds for any bounded and continuous function f on the real line.
7. Prove that every weakly compact set in a Banach space is a bounded set.
8. Let $\{x_n\}$ be a sequence in a normed space X such that $x_n \to x \in X$ weakly. Prove that there is a sequence $\{y_n\}$ such that each y_n belongs to the convex hull of $\{x_1, x_2, \ldots, x_n\}$ and $\|y_n - x\| \to 0$.
9. Let a and b be in $\mathcal{A}$. Show that $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$. Is $\sigma(ab) = \sigma(ba)$ always true? Prove or disprove your answer for the second question.
10. Show that the set G of invertible elements in $\mathcal{A}$ is open and the mapping $x \mapsto x^{-1}$ on G is continuous.

End of Paper