Department of Applied Mathematics, National Sun Yat-sen University
Ph.D. Program Qualifying Exam
Functional Analysis, Fall 2016

Answer any 5 of the following questions. (20 points for each question)

1. Show that the weak closure of the unit circle $\{x \in \mathfrak{X} : \|x\| = 1\}$ in an infinite-dimensional normed space $\mathfrak{X}$ is the closed unit ball $\{x \in \mathfrak{X} : \|x\| \leq 1\}$.

2. Suppose that u is a unitary element in a unital C^*-algebra $\mathcal{A}$ with spectrum $\sigma(u)$ strictly smaller than the unit circle $\mathbb{T}$. Show that there exists a self-adjoint element a in $\mathcal{A}$ such that $u = \exp(ia)$. Does the conclusion still hold if $\sigma(u) = \mathbb{T}$?

3. Let a be a self-adjoint element in a unital C^*-algebra $\mathcal{A}$ with $\|a\| \leq 1$. Show that there exists a self-adjoint element b in $\mathcal{A}$ such that $a = 2b(1 + b^2)^{-1}$.

4. Show that an element a in a unital C^*-algebra $\mathcal{A}$ is positive if and only if $\phi(a) \geq 0$ for every state ϕ on $\mathcal{A}$.

5. Let a be an invertible element in a Banach algebra $\mathcal{A}$. Show that an element $b \in \mathcal{A}$ satisfying $\|b - a\| \leq \|a^{-1}\|^{-1}$ is invertible.

6. Let $\mu, \mu_1, \mu_2, \ldots$ be Borel probability measures on $\mathbb{R}$. Show that μ_n converges to μ in the weak* topology if and only if

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi \, d\mu_n = \int_{\mathbb{R}} \varphi \, d\mu$$

   for every $\varphi \in C_c(\mathbb{R})$.

7. Let $T, S \in \mathcal{B}(\mathcal{H})$ satisfy $T^*T \leq S^*S$. Show that there exists $A \in \mathcal{B}(\mathcal{H})$ such that $\|A\| \leq 1$ and $T = AS$.

8. Show that every weakly compact set in a Banach space is a bounded set.

9. Let p and q be orthogonal projections of a complex Hilbert space onto closed subspaces $\mathcal{M}$ and $\mathcal{N}$, respectively. Show that $p+q$ is an orthogonal projection if and only if $p \perp q$.

End of Paper