Qualifying Examination (PhD)

Functional Analysis (Fall, 2013)

Answer any $\underline{5}$ of the following questions. (20 points each question)

1. (a) Let Y be a closed subspace of a Banach space X, and $x_0 \notin Y$. Show that there is a continuous linear functional $f \in X^*$ such that $\|f\| = 1$, $f_{|Y} \equiv 0$, and $f(x_0) = \operatorname{dist}(x_0, Y)$.

   (b) Recall that by a *hyperplane* of a Banach space X we mean any subspace Y of codimension 1 (that is, $\dim(X/Y) = 1$).

   Let Y be a subspace of a Banach space X. Show that Y is a closed hyperplane if and only if there is $f \in X^*$ such that $Y = f^{-1}(0)$.

2. Let X, Y be Banach spaces and $T \in B(X, Y)$. Show that the following are equivalent:

   (a) $T(X)$ is closed.

   (b) T is an open map when considered as a map from X onto $T(X)$.

   (c) There is $M > 0$ such that for every $y \in T(X)$ there is $x \in T^{-1}(y)$ satisfying $\|x\|_X \leq M\|y\|_Y$.

3. Let X, Y be Banach spaces, X reflexive, and $T \in B(X, Y)$. T is called a *completely continuous* operator, if for every weakly convergent sequence $\{x_n\}$ to x in X, $T(x_n)$ is convergent to $T(x)$ in Y. Show that if T is completely continuous, then it is compact.

4. (a) Let T be a diagonal operator on ℓ_2 associated with a bounded sequence of complex numbers $\{c_i\}$; that is, $T\big((x_i)\big) = (c_i x_i)$. Find the set of all eigenvalues of T and the spectrum of T.

   (b) Let K be a compact set in the scalar field. Show that there is an operator $T \in B(\ell_2)$ such that $\sigma(T) = K$.

5. (a) Show that if X is a finite-dimensional Banach space, then every linear functional f on X is continuous.

   (b) Show that if X is an infinite-dimensional Banach space, then X admits a discontinuous linear functional.

6. Show that every weakly compact set C in a Banach space is a bounded set.

7. Let H be a Hilbert space, and $A \in B(H)$. Show that $\|A\| = \|A^*\| = \|A^*A\|^{1/2}$.

End of Paper