Department of Applied Mathematics, National Syn Yat-sen University

Qualifying Examination (PhD)

Functional Analysis (Fall, 2012)

Answer any 5 of the following questions. (20 points each question)

1. Let X be a real normed space with norm $\|\cdot\|$ and X_c be the complexification $X \oplus iX = \{x + iy : x, y \in X\}$ of X with vector space operations defined by

$$(x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2), \quad x_1, x_2, y_1, y_2 \in X;$$
$$(\alpha + i\beta)(x + iy) = \alpha x - \beta y + i(\beta x + \alpha y), \quad x, y \in X, \text{ and } \alpha, \beta \in \mathbb{R}.$$

   (a) Show that the function $\|\cdot\|_c : X_c \to \mathbb{R}$ defined via the formula

$$\|z\|_c = \sup_{\theta \in [0, 2\pi]} \|x \cos\theta + y \sin\theta\|, \quad z = x + iy \in X_c,$$

   is a norm on X_c that extends $\|\cdot\|$.

   (b) Show that

$$\frac{1}{2}\left(\|x\| + \|y\|\right) \le \|z\|_c \le \|x\| + \|y\|$$

   for each $z = x + iy \in X_c$.

   (c) Show that if X is a Banach space, then X_c is also a Banach space.

2. Let $\sum x_i$ be a series in a Banach space X, and $x \in X$. We say that the series $\sum x_i$ is unconditionally convergent to x, if, for every $\epsilon > 0$, there is a finite set $F \subseteq \mathbb{N}$ such that $\left\|x - \sum_{i \in F'} x_i\right\| < \epsilon$ whenever F' is a finite set in $\mathbb{N}$ satisfying $F' \supset F$. And we say that $\sum x_i$ is unconditionally Cauchy if, given $\epsilon > 0$, there is a finite set F in $\mathbb{N}$ such that $\left\|\sum_{F'} x_i\right\| < \epsilon$ whenever F' is a finite set in $\mathbb{N}$ satisfying $F' \cap F = \emptyset$.

   Show that $\sum x_i$ is unconditionally Cauchy if and only if it is unconditionally convergent.

3. Let P_1 and P_2 be orthogonal projections on M_1 and M_2 on a complex Hilbert space H. We say that P_1 and P_1 are orthogonal, denoted by $P_1 \perp P_2$, if $M_1 \perp M_2$. Show that $P = P_1 + P_2$ is an orthogonal projection if and only if $P_1 \perp P_2$.

4. Show that a normed space X is a Banach space if and only if $\sum y_n$ converges whenever $\|y_n\| \le 2^{-n}$ for every n.

5. Let $\{x_i\}_{i=1}^n$ be a linearly independent set of vectors in a Banach space X and $\{\alpha_i\}_{i=1}^n$ be a finite set of real numbers. Show that there is $f \in X^*$ such that $f(x_i) = \alpha_i$ for $i = 1, 2, \ldots, n$.

6. Let X be a normed space with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that X in both of them is a complete space. Assume that $\|\cdot\|_1$ is not equivalent to $\|\cdot\|_2$. Let I_1 be the identity map from $(X, \|\cdot\|_1)$ onto $(X, \|\cdot\|_2)$ and I_2 be the identity map from $(X, \|\cdot\|_2)$ onto $(X, \|\cdot\|_1)$. Show that neither I_1 nor I_2 are continuous.

End of Paper