Department of Applied Mathematics, National Sun Yat-sen University

Qualifying Examination (PhD)

Functional Analysis (Spring, 2012)

Answer any <u>5</u> of the following questions.

1. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \quad \forall n \geq 1.$$

   Suppose $\sum_{n=1}^{\infty} \delta_n < +\infty$ and $\sum_{n=1}^{\infty} b_n < +\infty$.

   (a) Show that $\sum_{n=1}^{\infty} \log(1+\delta_n) < +\infty$.

   (b) Show that the infinite product $\prod_{n=1}^{\infty}(1+\delta_n)$ exists.

   (c) Show that $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence.

   (d) Show that $\lim_{n\to\infty} a_n$ exists.

2. Let C be a nonempty closed subset of a Banach space. Let $T : C \to C$ be a map such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

   If $0 \leq k < 1$, show that T has a fixed point z in C, i.e., $Tz = z$. What can you say if $k = 1$?

3. Show that every weak convergent sequence $\{x_n\}$ in a Banach space is norm bounded. In case all x_n are contained in a norm compact set, show that x_n converges in norm.

4. Let ℓ_1 be the Banach space of all absolutely summable seqences $x = \{x_n\}$ with norm $\|x\| := \sum_{n=1}^{\infty} |x_n| < +\infty$.

   (a) Show that every weak convergent sequence in ℓ_1 is norm convergent.

   (b) Show that the every topology of ℓ_1 does not equal the norm topology.

5. Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. Let $\varphi : H \times H \to \mathbb{C}$ be linear in the first variable and conjugate linear in the second variable. Suppose there are positive constants M, m such that

$$|\varphi(x, y)| \leq M\|x\|\|y\| \text{ and } |\varphi(x, x)| \geq m\|x\|^2, \quad \forall x, y \in H.$$

   Show that there exists a unique invertible bounded linear operator $A : H \to H$ such that

$$\varphi(x, y) = \langle x, Ay \rangle, \quad \forall x, y \in H,$$

   and

$$\|A^{-1}\| \leq \frac{1}{m}.$$

6. Let E be an infinite dimensional Banach space.

   (a) Prove that the range of a compact linear operator $T : E \to E$ must be separable.

   (b) Suppose $T : E \to E$ satisfies $I + 2T + 3T^2 + 4T^3 + 5T^4 = 0$. Show that T cannot be compact.

7. Let $C[0,1]$ be the Banach space of all continuous functions. Show that a sequence $f_n \to f$ weakly in $C[0,1]$ if and only if there is a positive number $M > 0$ such that

$$|f_n(x)| \leq M, \quad \forall x \in [0,1], \forall n = 1, 2, \ldots,$$

and

$$f_n(x) \to f(x), \quad \forall x \in [0,1].$$

End of Paper