Ph.D. Qualifying Examination

Functional Analysis National Sun Yat-sen University Feb 17, 2011

Answer any **5** of the following questions.

1. Let u be a semi-inner product on a linear space X and put $N = \{x \in X : u(x,x) = 0\}$.
   (a) Show that N is a linear subspace of X.
   (b) Show that if $\langle x+N, y+N\rangle \equiv u(x,y)$ for all $x+N$ and $y+N$ in the quotient space X/N, then $\langle\cdot,\cdot\rangle$ is a well-defined inner product on X/N.

2. Let X be a normed space. Prove that X is a Banach space if and only if whenever $\{x_n\}$ is a sequece in X such that $\sum\|x_n\| < \infty$, then $\sum_{n=1}^{\infty} x_n$ converges in X.

3. Let X be a vector space and suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X and that T_1 and T_2 are the corresponding topologies. Show that if X is complete in both norms and $T_1 \supseteq T_2$, then $T_1 = T_2$.

4. Let X be a normed space and let $\{x_n\}$ be a sequence in X such that $x_n \to x$ weakly. Show that there is a sequence $\{y_n\}$ such that $y_n \in \operatorname{co}\{x_1, x_2, \ldots, x_n\}$ and $\|y_n - x\| \to 0$.

5. Let A be a Banach algebra with identity, and $a, b \in A$. Show that $\sigma(ab)\cup\{0\} = \sigma(ba)\cup\{0\}$ and give an example such that $\sigma(ab) \neq \sigma(ba)$.

6. If H is a Hilbert space, A is a normal operator on H, and $f \in \operatorname{Hol}(A)$, show that $f(A)$ is normal.