Functional Analysis

Ph.D. Qualifying Examination

National Sun Yat-sen University

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Answer 5 out of the following 6 problems. Each problem carries 20%. The total is 100%. Unless otherwise stated, X denotes a Banach space over a scalar field K, and X^* denotes its dual.

(1). We say $x \perp y$ (x is orthogonal to y in X if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in K$.

(a) Show by an example that $x \perp y$ need not imply $y \perp x$.

(b) Show by an example that $x \perp y$ and $x \perp z$ need not imply $x \perp (y + z)$.

(c) Show that if X is a Hilbert space with inner product $(\cdot, \cdot)$, then $x \perp y$ if and only if $(x, y) = 0$.

(2). A sequence $A = \{x_n\}$ is said to be complete in X if the set of all finite linear combinations of elements of A is dense in X. Show that A is complete in X if and only if for any $f \in X^*$, $f(x_n) = 0$ for all $x_n \in A$ implies that $f \equiv 0$.

(3). Suppose that Y is also a Banach space, and $T: X \to Y$ be a compact linear operator. Show that if $\dim(X/Ker(T)) = \infty$, then there is a sequence $\{x_n\} \subset X$ such that $\|x_n\| \geq 1$ but $y_n = T(x_n) \to 0$.

(4). (a) Let $A \subset X$ such that for any fixed $f \in X^*$, $\sup_{x \in A} |f(x)| < \infty$. Show that $\sup_{x \in A} \|x\| < \infty$.

(b) Let f be a linear functional on a Hilbert space H with domain H. Show that f is continuous if and only if $Ker(f)$ is closed in H.

(5). (a) Show that a Banach space H is a Hilbert space if and only if the norm satisfies the parallelogram law:

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2), \qquad \text{for any } f, g \in H$$

(b) Show that $L^\infty[0,1]$, with its equipped norm, is a Banach space, but is not a Hilbert space.

(6). Let $\mathcal{A}$ be a Banach algebra with identity 1 and $x \in \mathcal{A}$.

(a) Show that if $\|x - 1\| < 1$, then x is invertible. What is the inverse of x?

(b) Hence show that the set G of invertible elements in $\mathcal{A}$ is open and the mapping $x \longmapsto x^{-1}$ on G is continuous.

End of Paper