Department of Applied Mathematics, National Sun Yat-sen University

Qualifying Examination (PhD)

FUNCTIONAL ANALYSIS (FALL, 2008)

Answer any <u>5</u> of the following questions.

1. Let N be the 2-dimensional real vector space $\mathbb{R}^2$. Sketch the unit ball of N in the xy-coordinate plane when N is equipped with the p-norm for $p = 1, 2$, and ∞, respectively. What happens when $p = 1/2$? Show that all three topologies induced by the metrics $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ coincide.

2. (a) Prove that a Banach space H is a Hilbert space if and only if the norm satisfies the Parallelogram Law:

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2), \quad \forall f, g \in H.$$

   (b) Prove that $C[0,1]$ is a Banach space under the supremum norm $\|\cdot\|_\infty$, but $C[0,1]$ is not a Hilbert space under the supremum norm $\|\cdot\|_\infty$.

3. Let A be a bounded linear operator on a complex Hilbert space H. Prove that the following statements are equivalent.

   (a) A is normal.

   (b) $\|Ah\| = \|A^*h\|$ for all h in H.

   (c) The real and the imaginary parts of A commute.

4. Let K be a non-empty closed convex subset of a Hilbert space H. Prove that K contains a unique vector of minimum norm. In general, we can define a map $P : H \to K$ such that $d(x, P(x)) = \inf\{d(x,k) : k \in K\}$. Show that P is well-defined and continuous in the norm topology.

5. Recall that a linear operator $T : E \to F$ between Banach spaces is called a compact operator if T sends the closed unit ball U_E of E to a relatively (norm) compact subset of F.

   (a) Show that every compact linear operator is norm continuous.

   (b) Show that every norm continuous linear operator of finite rank (i.e., $\dim TE < +\infty$) is compact.

   (c) Show that the set of all compact linear operators from a Banach space E into E forms a non-zero norm closed ideal of the algebra $B(E,E)$ of all norm continuous linear operators from E into E.

6. Show that all unital abelian C*-algebra A is isometrically *-isomorphic to the algebra $C(X)$ of continuous complex-valued functions defined on a compact Hausdorff space X. (Hint: You can start with the Gelfand transformation of abelian Banach algebra ....)