Functional Analysis

PHD-Qualified test, 2006

(1).Prove or disprove the following problems

(a) Let X be a normed linear space and X_0 a proper closed subspace of X. Then $\forall \theta, 0 < \theta < 1, \exists x_\theta \in X$ such that $\| x_\theta \| = 1$ and $\|x - x_\theta\| \geq \theta, \forall x \in X_\theta$. (10 points)

(b) Following the symbols of (a), If X_0 is finite dimensional then x_θ can be chosen distance 1 from X_0.(10 points)

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(2).Prove the following problems

(a) Let X and Y be normed linear spaces and $X_0 \neq \{0\}$. If $L(X, Y)$ is a Bnach space, then Y is a Banach space.(10 points)

(b) Let X be a TVS(Topological vector space) and $K \subseteq X$ absolute convex(ie. it is balance and convex) and absorbing. Then the Minkowski functional P_K of K(gauge of K) defined by $P(x) = \inf\{t > 0 : x \in tK\}$ is continuous if and only if K is a neighborhood of 0. (10 points)

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(3).Prove or disprove the following problems:

Let A be a C^* algebra, if a, b are positive elements of A.

(a) $a \leq b$ implies that $a^{\frac{1}{2}} \leq b^{\frac{1}{2}}$.(10 points)

(b) $a \leq b$ implies that $a^2 \leq b^2$. (10 points)

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(4).Prove that if H is a separable Hilbert space then the proper and closed two side ideal of $B(H)$ is $B_0(H)$(the space of compact operators)(20 points)

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(5).Prove that if H is a Hilbert space then the subset S of $B(H)$ which is linearly spanned by the rank one projections is dense in $B_0(H)$(the space of compact operators)(20 points).

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